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Matching NLO QCD computations with parton shower simulations: the POWHEG method

Stefano Frixione
INFN, Sezione di Genova,
Via Dodecaneso 33, 16146 Genova, Italy
E-mail: Stefano.Frixione@cern.ch

Paolo Nason
INFN, Sezione di Milano-Bicocca,
Piazza della Scienza 3, 20126 Milan, Italy
E-mail: Paolo.Nason@mib.infn.it

Carlo Oleari
Università di Milano-Bicocca and INFN, Sezione di Milano-Bicocca,
Piazza della Scienza 3, 20126 Milan, Italy
E-mail: Carlo.Oleari@mib.infn.it

Abstract: The aim of this work is to describe in detail the POWHEG method, first suggested by one of the authors, for interfacing parton-shower generators with NLO QCD computations. We describe the method in its full generality, and then specify its features in two subtraction frameworks for NLO calculations: the Catani-Seymour and the Frixione-Kunszt-Signer approach. Two examples are discussed in detail in both approaches: the production of hadrons in e^+e^- collisions, and the Drell-Yan vector-boson production in hadronic collisions.

Keywords: NLO Computations, QCD, Hadronic Colliders.

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1. Introduction

In the past two decades, next-to-leading order (NLO) QCD computations have become standard tools for phenomenological studies at lepton and hadron colliders. QCD tests have been mainly performed by comparing NLO results with experimental measurements, with the latter corrected for detector effects.

On the experimental side, leading order (LO) calculations, implemented in the context of general purpose Shower Monte Carlo (SMC) programs, have been the main tools used in the analysis. SMC programs include dominant QCD effects at the leading logarithmic level, but do not enforce NLO accuracy. These programs were routinely used to simulate background processes and signals in physics searches. When a precision measurement was needed, to be compared with an NLO calculation, one could not directly compare the experimental output with the SMC output, since the SMC does not have the required accuracy. The SMC output was used in this case to correct the measurement for detector effects, and the corrected result was compared to the NLO calculation.

In view of the positive experience with QCD tests at the NLO level, it has become clear that SMC programs should be improved, when possible, with NLO results. In this way a large amount of the acquired knowledge on QCD corrections would be made directly available to the experimentalists in a flexible form that they could easily use for simulations.

The problem of merging NLO calculations with parton shower simulations is basically that of avoiding overcounting, since the SMC programs do implement approximate NLO corrections already. Several proposals have appeared in the literature [1 − 4] that can be applied to both e^+e^- and hadronic collisions, and two approaches [5, 6] suitable for e^+e^- annihilation. Furthermore, proposals for new shower algorithms, that should be better suited for merging with NLO results, have appeared in the literature (see refs. [7 − 12]).

The `MC@NLO` proposal [2] was the first one to give an acceptable solution to the overcounting problem. The generality of the method has been explicitly proven by its application to processes of increasing complexity, such as heavy-flavour-pair [13] and single-top [14] production.[1] The basic idea of `MC@NLO` is that of avoiding the overcounting by subtracting from the exact NLO cross section its approximation, as implemented in the SMC program to which the NLO computation is matched. Such approximated cross section (which is the sum of what have been denoted in [2] as MC subtraction terms) is computed analytically, and is SMC dependent. On the other hand, the MC subtraction terms are process-independent, and thus, for a given SMC, can be computed once and for all. In the current version of the `MC@NLO` code, the MC subtraction terms have been computed for `HERWIG` [15]. In general, the exact NLO cross section minus the MC subtraction terms does not need to be positive. Therefore `MC@NLO` can generate events with negative weights. For the processes implemented so far, negative-weighted events are about 10–15% of the total. Their presence does not imply a negative cross section, since at the end physical distributions must turn out to be positive.

The features implemented in `MC@NLO` can be summarized as follows:

- Infrared-safe observables have NLO accuracy.
- Collinear emissions are summed at the leading-logarithmic level.
- The double logarithmic region (i.e. soft and collinear gluon emission) is treated correctly if the SMC code used for showering has this capability.

In the case of `HERWIG` this last requirement is satisfied, owing to the fact that its shower is based upon an angular-ordered branching.

In ref. [4] a method, to be called POWHEG in the following (for Positive Weight Hardest Emission Generator), was proposed that overcomes the problem of negative weighted events, and that is not SMC specific. In the POWHEG method the hardest radiation is generated first, with a technique that yields only positive-weighted events using the exact NLO matrix elements. The POWHEG output can then be interfaced to any SMC program that is either $p_{\rm T}$-ordered, or allows the implementation of a $p_{\rm T}$ veto.[2] However, when

[1] A complete list of processes implemented in `MC@NLO` can be found at http://www.hep.phy.cam.ac.uk/theory/webber/MCatNLO.

[2] All SMC programs compatible with the *Les Houches Interface for User Processes* [16] should comply with this requirement.

interfacing POWHEG to angular-ordered SMC programs, the double-log accuracy of the SMC is not sufficient to guarantee the double-log accuracy of the whole result. Some extra soft radiation (technically called vetoed-truncated shower in ref. [4]) must also be included in order to recover double-log accuracy. In fact, angular ordered SMC programs may generate soft radiation before generating the radiation with the largest $p_{\rm T}$, while POWHEG generates it first. When POWHEG is interfaced to shower programs that use transverse-momentum ordering, the double-log accuracy should be correctly retained if the SMC is double-log accurate. The `ARIADNE` program [17] and `PYTHIA` 6.4 [18] (when used with the new showering formalism), both adopt transverse-momentum ordering, in the framework of dipole-shower algorithm [19 − 21], and aim to have accurate soft resummation approaches, at least in the large N_c limit (where N_c is the number of colours).

A proof of concept for the POWHEG method has been given in ref. [22], for ZZ production in hadronic collisions. In ref. [23] the method was also applied to $Q\bar{Q}$ hadroproduction. Detailed comparisons have been carried out between the POWHEG and `MC@NLO` results, and reasonable agreement has been found, which nicely confirms the validity of both approaches. In ref. [24] the POWHEG method, interfaced to the `HERWIG++` Monte Carlo [25], has been applied to e^+e^- annihilation, and compared to LEP data. The method yields better fits compared to `HERWIG++` with matrix-element corrections. The authors of ref. [24] have also provided an estimate of the effects of the truncated shower, which turned out to be small.

In the present work we give a detailed description of the POWHEG method. Our aim is to provide all of the necessary formulae and procedures for its application to general NLO calculations. We first formulate POWHEG in a general subtraction scheme. Then, we illustrate it in detail in two such schemes: the Frixione, Kunszt and Signer (FKS) [26, 27] and the Catani and Seymour (CS) [28] one. The CS method has been widely used in the literature. On the other hand, the FKS method has already been used extensively in the `MC@NLO` implementations. Furthermore, the NLO cross sections for vector-boson and heavy-quark pair production used in the POWHEG implementations of refs. [22, 23] have a treatment of initial-state radiation that is essentially the same one used in FKS.

Our paper is organized as follows. In section 2 we summarize the general features of the NLO computations and of the subtraction formalisms. In section 2.4 all the details of the FKS subtraction method are given, and in section 2.5 the basic features of the CS approach are summarized.

In section 3 a general discussion of the inclusion of NLO corrections in a parton shower framework is given, together with a basic introduction of the POWHEG method.

In section 4 we go through all the details of the POWHEG method. The method is presented in general, and it is shown how to apply it within any subtraction framework. Thus, this section does not refer in particular to either the FKS or the CS method.

In section 4.5 we discuss the accuracy of the POWHEG approach in the resummation of soft-gluons effects. We show that, with an appropriate prescription for the evaluation of the running coupling used in POWHEG for the generation of radiation, one can easily obtain next-to-leading logarithmic (NLL) accuracy in soft-gluon radiation, provided the process in question has no more than three incoming or outgoing coloured partons at the Born

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level. If the Born process involves more than three coloured partons, there are left-over soft terms that are not correctly represented by POWHEG. We show, however, that with a modest modification of the algorithm one can also correctly resum these contributions at the level of the leading terms in a large N_c expansion.

In section 5 the formulae needed for the explicit construction of a POWHEG in the FKS framework is given. The same is done in section 6 for the CS method.

Finally, in section 7 two simple examples are discussed in both the FKS and the CS framework, namely the production of hadrons in e^+e^- annihilation, and the production of a massive vector boson (or a virtual photon) in hadronic collisions.

This paper is considerably long, and it involves many technical details. The length is partly due to the fact that we deal with two subtraction methods. The reader may skip the one she/he is not interested in. The example sections are particularly long and pedantic. The reader may not be interested in reading all of them. Section 4.5 is technically complex, but it may be almost completely skipped on a first reading.

2. NLO computations

2.1 Generalities

In this section, we describe the general features of an NLO calculation for a generic hadron-hadron collision process. In lepton-hadron and lepton-lepton collisions, the treatment is similar, but simpler. For example, in the case of lepton-lepton collisions, the parton-distribution functions for the incoming particles are replaced by delta functions.

We consider $2 \to n$ processes, where the momenta of the particles satisfy the momentum conservation

$$x_\oplus K_\oplus + x_\ominus K_\ominus = k_1 + \ldots + k_n \,, \tag{2.1}$$

where $x_{\oplus/\ominus}$ are the momentum fractions of the incoming partons, and $K_{\oplus/\ominus}$ the four-momenta of the incoming hadrons. In what follows, we also use the notation

$$k_\oplus = x_\oplus K_\oplus \,, \qquad\qquad k_\ominus = x_\ominus K_\ominus \,, \tag{2.2}$$

to denote the momenta of the incoming partons. We define, as usual,

$$S = (K_\oplus + K_\ominus)^2 \,, \qquad s = (k_\oplus + k_\ominus)^2 \,. \tag{2.3}$$

We denote by $\mathbf{\Phi}_n$ the set of variables

$$\mathbf{\Phi}_n = \{x_\oplus, x_\ominus, k_1, \ldots, k_n\} \tag{2.4}$$

constrained by momentum conservation (eq. (2.1)), and by the on-shell conditions for final-state particles. We collectively denote by $\mathcal{B}$ the squared matrix elements[3] relevant to the LO contributions to our process. The total cross section at leading order is given by

$$\sigma_{\rm LO} = \int d\mathbf{\Phi}_n \, \mathcal{L} \, \mathcal{B}(\mathbf{\Phi}_n) \tag{2.5}$$

[3] We always assume spin and colour sums and averages when needed, and the inclusion of the appropriate flux factor.

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where $\mathcal{L}$ is the parton luminosity[4]

$$\mathcal{L} = \mathcal{L}(x_\oplus, x_\ominus) = f_\oplus(x_\oplus)\, f_\ominus(x_\ominus)\,, \tag{2.6}$$

and

$$d\mathbf{\Phi}_n = dx_\oplus\, dx_\ominus\, d\Phi_n\left(k_\oplus + k_\ominus; k_1, \ldots, k_n\right)\,, \tag{2.7}$$

with $d\Phi_n$ the n-body phase space

$$d\Phi_n\left(q; k_1, \ldots, k_n\right) = (2\pi)^4\, \delta^4\left(q - \sum_{i=1}^{n} k_i\right) \prod_{i=1}^{n} \frac{d^3 k_i}{(2\pi)^3\, 2k_i^0}\,. \tag{2.8}$$

In case of leptons in the initial state, the corresponding parton distribution function $f(x)$ in eq. (2.6) is replaced by $\delta(1-x)$.

The real contributions at the NLO arise from the tree-level squared amplitudes for the $2 \to n+1$ parton process, which we denote by $\mathcal{R}$. As before, we denote by $\mathbf{\Phi}_{n+1}$ the corresponding set of variables

$$\mathbf{\Phi}_{n+1} = \{x_\oplus, x_\ominus, k_1, \ldots, k_{n+1}\} \tag{2.9}$$

constrained by momentum conservation and on-shell conditions.

The virtual contributions arise from the interference of the one-loop amplitudes times the LO amplitudes. We denote by $\mathcal{V}_{\rm b}$ the renormalized virtual corrections, that is, we assume that all ultraviolet divergences have already been removed by renormalization. These terms still contain infrared divergences. Therefore, they are computed in $d = 4 - 2\epsilon$ dimensions, and the divergences appear as $1/\epsilon^2$ and $1/\epsilon$ poles. The subscript b (for "bare") reminds us of the presence of infrared divergences in the amplitude.

In hadronic collisions, the complete cancellation of the initial-state collinear singularities is achieved by adding two counterterms, one for each of the incoming partons ($\oplus$, $\ominus$), to the differential cross section. We denote them by $\mathcal{G}_{\oplus,{\rm b}}$ and $\mathcal{G}_{\ominus,{\rm b}}$. The factorization counterterms are infrared divergent in four dimensions. Therefore, they are computed in $d = 4 - 2\epsilon$ dimensions, and the divergences appear as $1/\epsilon$ poles. To remind this fact, also in this case a subscript b has been included in the notation.

The total NLO cross section is given by[5]

$$\begin{aligned} \sigma_{\rm NLO} = & \int d\mathbf{\Phi}_n\, \mathcal{L}\left[\mathcal{B}(\mathbf{\Phi}_n) + \mathcal{V}_{\rm b}(\mathbf{\Phi}_n)\right] + \int d\mathbf{\Phi}_{n+1}\, \mathcal{L}\, \mathcal{R}(\mathbf{\Phi}_{n+1}) \\ & + \int d\mathbf{\Phi}_{n,\oplus}\, \mathcal{L}\, \mathcal{G}_{\oplus,{\rm b}}(\mathbf{\Phi}_{n,\oplus}) + \int d\mathbf{\Phi}_{n,\ominus}\, \mathcal{L}\, \mathcal{G}_{\ominus,{\rm b}}(\mathbf{\Phi}_{n,\ominus})\,, \end{aligned} \tag{2.10}$$

where

$$d\mathbf{\Phi}_{n+1} = dx_\oplus\, dx_\ominus\, d\Phi_{n+1}\left(k_\oplus + k_\ominus; k_1, \ldots, k_{n+1}\right)\,. \tag{2.11}$$

[4]In this section we drop the parton flavours and the scale dependence in the luminosity, for ease of notation.

[5]The $\mathcal{G}_{\oplus,{\rm b}}$ terms are present only for incoming hadrons. If one or both the incoming particles are leptons, the corresponding $\mathcal{G}_{\rm b}$ is zero.

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$\mathbf{\Phi}_{n,\oplus\!\!\!\ominus}$ denotes configurations in which one of the final-state partons is collinear to one of the incoming partons. Thus, such configurations are effectively n-body final-state ones, except for the energy degree of freedom of the parton collinear to the beam. We then write

$$\mathbf{\Phi}_{n,\oplus} = \{x_\oplus, x_\ominus, z, k_1, \ldots, k_n\} \ , \quad z\, x_\oplus K_\oplus + x_\ominus K_\ominus = \sum_{i=1}^{n} k_i \ , \tag{2.12}$$

$$\mathbf{\Phi}_{n,\ominus} = \{x_\oplus, x_\ominus, z, k_1, \ldots, k_n\} \ , \quad x_\oplus K_\oplus + z\, x_\ominus K_\ominus = \sum_{i=1}^{n} k_i \ , \tag{2.13}$$

where z is the fraction of momentum of the incoming parton after radiation. We can associate with the phase-space configuration $\mathbf{\Phi}_{n,\oplus\!\!\!\ominus}$ an *underlying n-body configuration* $\bar{\mathbf{\Phi}}_n$ defined as

$$\bar{\mathbf{\Phi}}_n = \{\bar{x}_\oplus, \bar{x}_\ominus, k_1, \ldots k_n\} \, , \qquad \bar{x}_{\oplus\!\!\!\ominus} = z x_{\oplus\!\!\!\ominus} \, , \qquad \bar{x}_{\ominus\!\!\!\oplus} = x_{\ominus\!\!\!\oplus} \ . \tag{2.14}$$

Thus, the values of $\bar{x}_{\oplus\!\!\!\ominus}$ in the underlying n-body configuration are constrained by momentum conservation, and do not depend upon z. We also define

$$d\mathbf{\Phi}_{n,\oplus} = dx_\oplus\, dx_\ominus\, dz\, d\Phi_n \left(z\, k_\oplus + k_\ominus; k_1, \ldots, k_n\right) \ , \tag{2.15}$$

$$d\mathbf{\Phi}_{n,\ominus} = dx_\oplus\, dx_\ominus\, dz\, d\Phi_n \left(k_\oplus + z\, k_\ominus; k_1, \ldots, k_n\right) \ . \tag{2.16}$$

We now consider a generic observable O, function of the final-state momenta. O could be, for example, a product of theta functions describing a particular histogram bin for the distribution of some kinematic observable. Its expected value is given by

$$\begin{aligned}\langle O \rangle = &\int d\mathbf{\Phi}_n\, \mathcal{L}\, O_n(\mathbf{\Phi}_n) \left[\mathcal{B}(\mathbf{\Phi}_n) + \mathcal{V}_{\mathrm{b}}(\mathbf{\Phi}_n)\right] \\ &+ \int d\mathbf{\Phi}_{n+1}\, \mathcal{L}\, O_{n+1}(\mathbf{\Phi}_{n+1})\ \mathcal{R}(\mathbf{\Phi}_{n+1}) \\ &+ \int d\mathbf{\Phi}_{n,\oplus}\, \mathcal{L}\, O_n\big(\bar{\mathbf{\Phi}}_n\big)\ \mathcal{G}_{\oplus,\mathrm{b}}(\mathbf{\Phi}_{n,\oplus}) + \int d\mathbf{\Phi}_{n,\ominus}\, \mathcal{L}\, O_n\big(\bar{\mathbf{\Phi}}_n\big)\ \mathcal{G}_{\ominus,\mathrm{b}}(\mathbf{\Phi}_{n,\ominus}) \, ,\end{aligned} \tag{2.17}$$

where O_n and O_{n+1} are the expressions of the observable O in terms of n and $(n+1)$ final-state particle momenta, and $\mathbf{\Phi}_n$, in the $\mathbf{\Phi}_{n,\oplus\!\!\!\ominus}$ integrals, is the corresponding underlying n-body configuration. We require that O is an infrared-safe observable, and, furthermore, we require that the Born contribution in eq. (2.17) (i.e. the term proportional to $\mathcal{B}$) is infrared finite (thus, for example, if our n-body process corresponds to $Z+\text{jet}$ production, the observable O_n must suppress the region where the jet is emitted at low transverse momentum). Under these assumptions, the real matrix elements contribution (i.e. the term proportional to $\mathcal{R}$) is finite in the whole phase space $d\Phi_{n+1}$, except for the regions that correspond to soft and collinear emissions. There, the divergences are integrable only in d dimensions, and yield $1/\epsilon^2$ and $1/\epsilon$ poles. Furthermore the divergences of each term on the r.h.s. of eq. (2.17) cancel in the sum, and the total cross section is finite. Observe that the argument of O in the last two terms on the right hand side of eq. (2.17) is set equal to $\bar{\mathbf{\Phi}}_n$ rather than $\mathbf{\Phi}_{n,\oplus\!\!\!\ominus}$, owing to the fact that O is an infrared-safe observable. The integrals in eq. (2.17) are usually too difficult to be performed analytically (because of the involved functional form of O) and, being divergent, they are not suited for numerical computations. For these reasons, different strategies have been proposed for the computation of observables in QCD. One of the most successful is the so-called subtraction method, pioneered in ref. [29], which we discuss in the next section.

2.2 Subtraction formalism

The subtraction formalism requires the definition of a set of functions $\mathcal{C}^{(\alpha)}$, called real counterterms. Each α is associated with a particular singular region, i.e. with a configuration that has either a final-state parton with four momentum equal to zero, or a final-state massless parton with momentum proportional to an initial-state or to another final-state massless parton. Furthermore, for each α, a mapping[6] $\mathbf{M}^{(\alpha)}$

$$\tilde{\mathbf{\Phi}}^{(\alpha)}_{n+1} = \mathbf{M}^{(\alpha)}\left(\mathbf{\Phi}_{n+1}\right)\ ,\quad \tilde{\mathbf{\Phi}}^{(\alpha)}_{n+1} = \left\{\tilde{x}^{(\alpha)}_{\oplus}, \tilde{x}^{(\alpha)}_{\ominus}, \tilde{k}^{(\alpha)}_{1}, \ldots, \tilde{k}^{(\alpha)}_{n+1}\right\} \tag{2.18}$$

is defined that maps the $(n+1)$-body configuration into a singular one.

The real counterterms and the mapping have the following property: for any infrared-safe observable $O_{n+1}(\mathbf{\Phi}_{n+1})$, that vanishes fast enough if $\mathbf{\Phi}_{n+1}$ approaches two singular regions at the same time, the function

$$\mathcal{R}(\mathbf{\Phi}_{n+1})\, O_{n+1}(\mathbf{\Phi}_{n+1}) - \sum_{\alpha} \mathcal{C}^{(\alpha)}(\mathbf{\Phi}_{n+1})\, O_{n+1}\left(\mathbf{M}^{(\alpha)}\left(\mathbf{\Phi}_{n+1}\right)\right) \tag{2.19}$$

has at most integrable singularities in the $\mathbf{\Phi}_{n+1}$ space. Observe that the above condition does not always imply that

$$\mathcal{R}(\mathbf{\Phi}_{n+1}) - \sum_{\alpha} \mathcal{C}^{(\alpha)}(\mathbf{\Phi}_{n+1}) \tag{2.20}$$

is also integrable. This is the case if the corresponding n-body process has no singularities, like, for example, in Z production in hadronic collisions.

Each singular region α is characterized by a different mapping, and, for this reason, we use the superscript α on the tilded variables. For ease of notation, we use the following *context convention*: if an expression is enclosed in the subscripted squared brackets

$$\left[\ldots\right]_{\alpha}, \tag{2.21}$$

we mean that all variables appearing inside have, when applicable, the superscripts corresponding to the subscript of the bracket. Thus we write

$$\tilde{\mathbf{\Phi}}^{(\alpha)}_{n+1} = \left[\left\{\tilde{x}_{\oplus}, \tilde{x}_{\ominus}, \tilde{k}_{1}, \ldots, \tilde{k}_{n+1}\right\}\right]_{\alpha}. \tag{2.22}$$

The form of the singular configurations $\tilde{\mathbf{\Phi}}^{(\alpha)}_{n+1}$ differs according to the nature of the singular region. More specifically:

[6] In some approaches, the counterterms are different from zero only in a finite neighborhood of the corresponding singular regions. In these cases, the mapping needs to be defined only there.

- If α is associated with a soft (S) region, the singular configuration has a final-state parton with null four-momentum.

- If α is associated with a final-state collinear singularity (FSC), the singular configuration has two massless final-state partons with parallel three-momenta.

- If α is associated with an initial-state collinear (ISC) singularity, the singular configuration has a massless outgoing parton with three-momentum parallel to the momentum of one incoming parton.

The mapping (2.18) must be smooth near the singular region, and it must become the identity there. In other words if, for instance, α is associated with the FSC region where the particles i and j become collinear, we must have $\tilde{\mathbf{\Phi}}_{n+1}^{(\alpha)} = \mathbf{\Phi}_{n+1}$ for $\vec{k}_i \parallel \vec{k}_j$. Notice that also the x's of the configurations $\tilde{\mathbf{\Phi}}_{n+1}^{(\alpha)}$ do not necessarily coincide with $x_\oplus$ and $x_\ominus$ for all $(n+1)$-body configurations $\mathbf{\Phi}_{n+1}$. On the other hand, they do coincide in the singular limit.

As in the case of the collinear configurations (eqs. (2.12) and (2.13)), we associate with each $\tilde{\mathbf{\Phi}}_{n+1}^{(\alpha)}$ configuration an n-body configuration $\bar{\mathbf{\Phi}}_n^{(\alpha)}$, that we will call the *underlying n-body configuration*

$$\bar{\mathbf{\Phi}}_n^{(\alpha)} = \left[\left\{\bar{x}_\oplus, \bar{x}_\ominus, \bar{k}_1, \ldots, \bar{k}_n\right\}\right]_\alpha \ . \tag{2.23}$$

$\bar{\mathbf{\Phi}}_n^{(\alpha)}$ is obtained as follows:

- If $\alpha \in \mathrm{S}$ (i.e. it is a soft region), $\bar{\mathbf{\Phi}}_n^{(\alpha)}$ is obtained by deleting the zero momentum parton.

- If $\alpha \in \mathrm{FSC}$ (i.e. it is a final-state collinear region), $\bar{\mathbf{\Phi}}_n^{(\alpha)}$ is obtained by replacing the momenta of the two collinear partons with their sum.

- If $\alpha \in \mathrm{ISC}$ (i.e. it is an initial state collinear region), $\bar{\mathbf{\Phi}}_n^{(\alpha)}$ is obtained by deleting the radiated collinear parton, and by replacing the momentum fraction of the initial-state radiating parton with its momentum fraction after radiation.

In all the above cases, the final-state momenta are relabelled with an index that takes values in the range $1, \ldots, n$. Observe that, as a consequence of the procedure itemized above, the variables in $\bar{\mathbf{\Phi}}_n^{(\alpha)}$ are constrained by momentum conservation

$$\bar{x}_\oplus K_\oplus + \bar{x}_\ominus K_\ominus = \sum_{j=1}^{n} \bar{k}_j \ . \tag{2.24}$$

Furthermore, for S or FSC regions, we have

$$\bar{x}_{\stackrel{\oplus}{\ominus}} = \tilde{x}_{\stackrel{\oplus}{\ominus}} \, . \tag{2.25}$$

This does not hold for ISC regions: in the $\oplus$ direction, for example, we have

$$\bar{x}_\oplus < \tilde{x}_\oplus \ , \qquad \bar{x}_\ominus = \tilde{x}_\ominus \ , \tag{2.26}$$

and the analogous one for the case of ISC in the $\ominus$ direction.

We stress that the difference in our notation between the $\tilde{\mathbf{\Phi}}_{n+1}^{(\alpha)}$ and $\bar{\mathbf{\Phi}}_{n}^{(\alpha)}$ is a minor one: the former has an unresolved parton, while in the latter all partons are resolved. On the other hand, it is necessary to introduce $\bar{\mathbf{\Phi}}_{n}^{(\alpha)}$ (together with the concept of underlying n-body configuration), since it is formally the argument of Born-like matrix elements, and will play a central role in the development of the POWHEG formalism.

In the subtraction method one rewrites the contribution to any observable O coming from real radiation in the following way

$$\int d\mathbf{\Phi}_{n+1}\,\mathcal{L}\,O_{n+1}(\mathbf{\Phi}_{n+1})\,\mathcal{R}(\mathbf{\Phi}_{n+1}) = \sum_{\alpha}\int d\mathbf{\Phi}_{n+1}\left[\tilde{\mathcal{L}}\,O_n(\bar{\mathbf{\Phi}}_n)\,\mathcal{C}(\mathbf{\Phi}_{n+1})\right]_{\alpha} + \int d\mathbf{\Phi}_{n+1}\left\{\mathcal{L}\,O_{n+1}(\mathbf{\Phi}_{n+1})\,\mathcal{R}(\mathbf{\Phi}_{n+1}) - \sum_{\alpha}\left[\tilde{\mathcal{L}}\,O_n(\bar{\mathbf{\Phi}}_n)\,\mathcal{C}(\mathbf{\Phi}_{n+1})\right]_{\alpha}\right\}, \tag{2.27}$$

where $\tilde{\mathcal{L}} = \mathcal{L}(\tilde{x}_{\oplus}, \tilde{x}_{\ominus})$. In this way, under the assumptions we have made about the counterterms, and the assumption that O is an infrared-safe observable, the second term on the r.h.s. of eq. (2.27) is integrable in $d = 4$ dimensions.

The first term on the r.h.s. of eq. (2.27) is divergent. In order to deal with it, we introduce, for each α, the $(n+1)$-phase space parametrization

$$\mathbf{\Phi}_{n+1} \overset{(\alpha)}{\Longleftrightarrow} \left\{\bar{\mathbf{\Phi}}_n^{(\alpha)}, \Phi_{\rm rad}^{(\alpha)}\right\}, \tag{2.28}$$

and the corresponding phase-space element

$$d\mathbf{\Phi}_{n+1} = d\bar{\mathbf{\Phi}}_n^{(\alpha)}\, d\Phi_{\rm rad}^{(\alpha)}. \tag{2.29}$$

In words, we parametrize the $(n+1)$-phase space in terms of an n-body phase space (obtained as described earlier), plus (three) more variables that describe the radiation process. The left-right arrow in eq. (2.28) indicates that the correspondence is one to one.[7] The range of the radiation variables in $\Phi_{\rm rad}^{(\alpha)}$ may depend upon $\bar{\mathbf{\Phi}}_n^{(\alpha)}$. Furthermore, eq. (2.29) implicitly defines a Jacobian, possibly dependent upon $\bar{\mathbf{\Phi}}_n^{(\alpha)}$, that we conventionally include into $d\Phi_{\rm rad}^{(\alpha)}$. We call the parametrization (2.28) the *emission factorization.*

We now distinguish two cases: the FSC+S case and the ISC one. In the former case we have

$$\tilde{\mathcal{L}} = \mathcal{L}(\tilde{x}_{\oplus}, \tilde{x}_{\ominus}) = \mathcal{L}(\bar{x}_{\oplus}, \bar{x}_{\ominus}). \tag{2.30}$$

Defining

$$\left[\bar{\mathcal{C}}\left(\bar{\mathbf{\Phi}}_n\right) = \int d\Phi_{\rm rad}\,\mathcal{C}(\mathbf{\Phi}_{n+1})\right]_{\alpha\in\{\rm FSC,S\}}, \tag{2.31}$$

we can write the generic term in the first sum on the r.h.s. of eq. (2.27) as follows

$$\left[\int d\mathbf{\Phi}_{n+1}\,\tilde{\mathcal{L}}\,O_n(\bar{\mathbf{\Phi}}_n)\,\mathcal{C}(\mathbf{\Phi}_{n+1}) = \int d\bar{\mathbf{\Phi}}_n\,\tilde{\mathcal{L}}\,O_n(\bar{\mathbf{\Phi}}_n)\,\bar{\mathcal{C}}\left(\bar{\mathbf{\Phi}}_n\right)\right]_{\alpha\in\{\rm FSC,S\}}. \tag{2.32}$$

[7]The correspondence needs only to be defined where the corresponding counterterm is non-vanishing.

In the ISC case, we cannot factor out the luminosity so easily, since $\tilde{x}_{\oplus\ominus} \neq \bar{x}_{\oplus\ominus}$. We define

$$\left[\bar{\mathcal{C}}\left(\bar{\mathbf{\Phi}}_n, z\right) = \int d\Phi_{\mathrm{rad}}\, \mathcal{C}(\mathbf{\Phi}_{n+1})\;\, z\, \delta\left(z - \bar{x}_{\oplus\ominus}/\tilde{x}_{\oplus\ominus}\right)\right]_{\alpha \in \left\{\mathrm{ISC}_{\oplus\ominus}\right\}}, \tag{2.33}$$

which formally introduces the momentum fraction z, and write

$$\left[\int d\mathbf{\Phi}_{n+1}\, \tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\, \mathcal{C}(\mathbf{\Phi}_{n+1}) = \int d\bar{\mathbf{\Phi}}_n \frac{dz}{z} \tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right) \bar{\mathcal{C}}\left(\bar{\mathbf{\Phi}}_n, z\right)\right]_{\alpha\in\{\mathrm{ISC}\}}. \tag{2.34}$$

Notice that, owing to the delta function in eq. (2.33) we have

$$\tilde{\mathcal{L}} = \mathcal{L}(\tilde{x}_\oplus, \tilde{x}_\ominus) = \begin{cases} \mathcal{L}(\bar{x}_\oplus/z, \bar{x}_\ominus) & \text{for} \quad \alpha \in \mathrm{ISC}_\oplus \\ \mathcal{L}(\bar{x}_\oplus, \bar{x}_\ominus/z) & \text{for} \quad \alpha \in \mathrm{ISC}_\ominus \end{cases}. \tag{2.35}$$

We also notice that the variables $\left\{\tilde{x}_\oplus, \tilde{x}_\ominus, z, \bar{k}_1, \ldots, \bar{k}_n\right\}$ in the ISC regions can be identified with the $\mathbf{\Phi}_{n,\oplus\ominus}$ variables in eqs. (2.12) and (2.13). In fact, the $\bar{k}_i$ are integration variables, and can be identify with the k_i's in eqs. (2.12) and (2.13). Furthermore, the $\tilde{x}_{\oplus\ominus}$ variables are identical to the $x_{\oplus\ominus}$ in eqs. (2.12) and (2.13), since those equations refer to a singular configuration, and (as we have remarked earlier) the mapping of eq. (2.18) is the identity in the singular region. It follows that the z variables of eqs. (2.14) and (2.33) are identical. Hence, from eqs. (2.15) and (2.16), we obtain

$$d\mathbf{\Phi}_{n,\oplus\ominus} = d\bar{\mathbf{\Phi}}_n \frac{dz}{z} \tag{2.36}$$

(the $1/z$ factor being due to the Jacobian for the change of variables $\bar{x}_{\oplus\ominus} \to \tilde{x}_{\oplus\ominus}$).

The choice of the counterterms in eq. (2.19) and of the mapping (2.18) should be such that the integrals in eqs. (2.31) and (2.33) are easily performed analytically in d dimensions. In this way, the $\bar{\mathcal{C}}$ terms contain explicitly the divergences as poles in ϵ.

We now write eq. (2.17) as

$$\begin{aligned} \langle O \rangle = & \int d\mathbf{\Phi}_n\, \mathcal{L}\, O_n(\mathbf{\Phi}_n)\, \left[\mathcal{B}(\mathbf{\Phi}_n) + \mathcal{V}_{\mathrm{b}}(\mathbf{\Phi}_n)\right] \\ & + \int d\mathbf{\Phi}_{n+1} \left\{\mathcal{L}\, O_{n+1}(\mathbf{\Phi}_{n+1})\, \mathcal{R}(\mathbf{\Phi}_{n+1}) - \sum_\alpha \left[\tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\, \mathcal{C}(\mathbf{\Phi}_{n+1})\right]_\alpha\right\} \\ & + \sum_{\alpha\in\{\mathrm{FSC,S}\}} \left[\int d\bar{\mathbf{\Phi}}_n\, \tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\, \bar{\mathcal{C}}\left(\bar{\mathbf{\Phi}}_n\right)\right]_\alpha + \sum_{\alpha\in\left\{\mathrm{ISC}_{\oplus\ominus}\right\}} \left[\int d\mathbf{\Phi}_{n,\oplus\ominus}\, \tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\, \bar{\mathcal{C}}\left(\mathbf{\Phi}_{n,\oplus\ominus}\right)\right]_\alpha \\ & + \int d\mathbf{\Phi}_{n,\oplus}\, \tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\, \mathcal{G}_{\oplus,\mathrm{b}}(\mathbf{\Phi}_{n,\oplus}) + \int d\mathbf{\Phi}_{n,\ominus}\, \tilde{\mathcal{L}}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\, \mathcal{G}_{\ominus,\mathrm{b}}(\mathbf{\Phi}_{n,\ominus})\,. \end{aligned} \tag{2.37}$$

Notice that, in the last line, we have substituted $\mathcal{L} \to \tilde{\mathcal{L}}$ for uniformity of notation. This is correct, since, as pointed out earlier, in the phase space of the collinear counterterms we have $x_{\oplus\ominus} = \tilde{x}_{\oplus\ominus}$.

It turns out that it is always possible to write

$$\mathcal{G}_{\oplus\ominus,\mathrm{b}}(\mathbf{\Phi}_{n,\oplus\ominus}) + \sum_{\alpha\in\left\{\mathrm{ISC}_{\oplus\ominus}\right\}} \bar{\mathcal{C}}^{(\alpha)}(\mathbf{\Phi}_{n,\oplus\ominus}) = \mathcal{G}_{\oplus\ominus}(\mathbf{\Phi}_{n,\oplus\ominus}) + \delta(1-z)\, \mathcal{G}^{\mathrm{div}}_{\oplus\ominus}\left(\bar{\mathbf{\Phi}}_n\right), \tag{2.38}$$

where $\mathcal{G}_{\oplus\ominus}(\mathbf{\Phi}_{n,\oplus\ominus})$ is finite in $d = 4$ dimensions.[8] The only remaining poles in ϵ are included

[8]We point out that $\mathcal{G}_{\oplus\ominus}$, although finite, may contain distributions associated with the soft region $z \to 1$.

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in the last term of eq. (2.38), and have soft origin. It also turns out that in the quantity

$$\mathcal{V}(\mathbf{\Phi}_n) = \mathcal{V}_{\rm b}(\mathbf{\Phi}_n) + \left[\sum_{\alpha\in\{\rm FSC,S\}} \bar{\mathcal{C}}^{(\alpha)}\left(\bar{\mathbf{\Phi}}_n\right) + \mathcal{G}^{\rm div}_{\oplus}\left(\bar{\mathbf{\Phi}}_n\right) + \mathcal{G}^{\rm div}_{\ominus}\left(\bar{\mathbf{\Phi}}_n\right) \right]^{\bar{\mathbf{\Phi}}_n=\mathbf{\Phi}_n}, \tag{2.39}$$

all poles in ϵ cancel. With the notation

$$\left[\ldots\right]^{\bar{\mathbf{\Phi}}_n=\mathbf{\Phi}_n}, \tag{2.40}$$

we mean that the argument between the brackets is evaluated for values of the phase-space variables $\bar{\mathbf{\Phi}}_n$ equal to $\mathbf{\Phi}_n$. Notice that the identification $\bar{\mathbf{\Phi}}_n = \mathbf{\Phi}_n$ is possible, since $\bar{\mathbf{\Phi}}_n$ refers to the underlying n-body configuration, that must correspond to the Born term. Defining now the following abbreviations

$$R = \mathcal{L}\,\mathcal{R}, \qquad C^{(\alpha)} = \tilde{\mathcal{L}}^{(\alpha)}\,\mathcal{C}^{(\alpha)}, \qquad G_{\oplus} = \tilde{\mathcal{L}}\,\mathcal{G}_{\oplus}, \qquad B = \mathcal{L}\,\mathcal{B}, \qquad V = \mathcal{L}\,\mathcal{V}\,, \tag{2.41}$$

equation (2.37) becomes

$$\begin{aligned}\langle O\rangle = &\int d\mathbf{\Phi}_n\, O_n(\mathbf{\Phi}_n)\,\Big[B(\mathbf{\Phi}_n) + V(\mathbf{\Phi}_n)\Big] \\ &+\int d\mathbf{\Phi}_{n+1}\left\{O_{n+1}(\mathbf{\Phi}_{n+1})\; R(\mathbf{\Phi}_{n+1}) - \sum_\alpha \left[O_n\left(\bar{\mathbf{\Phi}}_n\right)\; C(\mathbf{\Phi}_{n+1})\right]_\alpha\right\} \\ &+\int d\mathbf{\Phi}_{n,\oplus}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\; G_\oplus(\mathbf{\Phi}_{n,\oplus}) + \int d\mathbf{\Phi}_{n,\ominus}\, O_n\left(\bar{\mathbf{\Phi}}_n\right)\; G_\ominus(\mathbf{\Phi}_{n,\ominus})\ ,\end{aligned} \tag{2.42}$$

and it is now suited to be integrated numerically, since all the integrals that appear in it are finite and can be evaluated in 4 dimensions.

2.3 Subtraction formalism using the “plus” distributions

The subtraction method naturally arises when results of NLO computations are expressed in terms of distributions in final-state variables. In order to illustrate this issue, we assume now for simplicity that there is just one singular region, and, in 4 dimensions, we describe the kinematics of the emitted parton (with momentum k) in the centre-of-mass (CM) frame of the incoming partons with the following variables

$$\xi = 2k^0/\sqrt{s}\,, \qquad y = \cos\theta\,, \qquad \phi\,, \tag{2.43}$$

where $s = (k_\oplus + k_\ominus)^2$, θ is the angle of the emitted parton relative to a reference direction (typically another parton), and ϕ is an azimuthal variable around the same reference direction. The singular regions (soft and collinear) are associated with $\xi \to 0$ and $y \to 1$ respectively. More generally, in $d = 4 - 2\epsilon$ dimensions, we can write

$$\frac{d^{d-1}k}{2k^0(2\pi)^{d-1}} = \frac{s^{1-\epsilon}}{(4\pi)^{d-1}}\;\xi^{1-2\epsilon}\,(1-y^2)^{-\epsilon}\;d\xi\,dy\,d\Omega^{d-2}\,, \tag{2.44}$$

where

$$d\Omega^{d-2} = (\sin\phi)^{-2\epsilon}\,d\phi\,d\Omega^{d-3}\,, \qquad \int d\Omega^{d-3} = \frac{2\pi^{\frac{d-3}{2}}}{\Gamma\left(\frac{d-3}{2}\right)}\,. \tag{2.45}$$

If we write

$$\mathcal{R} = \frac{1}{\xi^2}\frac{1}{1-y}\left[\xi^2\,(1-y)\,\mathcal{R}\right]\,, \tag{2.46}$$

then $[\xi^2\,(1-y)\,\mathcal{R}]$ is regular for $\xi \to 0$ and $y \to 1$. The phase-space integral of $\mathcal{R}$ is infrared divergent, and in $d = 4-2\epsilon$ dimensions (with $\epsilon < 0$) the singular part of the integration is proportional to (see eq. (2.44))

$$\int_{-1}^{1} dy\,(1-y)^{-1-\epsilon}\int_0^1 d\xi\,\xi^{-1-2\epsilon}\,[\xi^2\,(1-y)\,\mathcal{R}]\,. \tag{2.47}$$

In order to deal with the singularities, one uses the expansions

$$\xi^{-1-2\epsilon} = -\frac{1}{2\epsilon}\delta(\xi) + \left(\frac{1}{\xi}\right)_+ - 2\epsilon\left(\frac{\log\xi}{\xi}\right)_+ + \mathcal{O}(\epsilon^2)\,, \tag{2.48}$$

$$(1-y)^{-1-\epsilon} = -\frac{2^{-\epsilon}}{\epsilon}\delta(1-y) + \left(\frac{1}{1-y}\right)_+ + \mathcal{O}(\epsilon)\,, \tag{2.49}$$

with the usual definition of the + prescription

$$\int_0^1 d\xi\,\left(\frac{1}{\xi}\right)_+ f(\xi) = \int_0^1 d\xi\,\frac{f(\xi)-f(0)}{\xi}\,, \tag{2.50}$$

$$\int_0^1 d\xi\,\left(\frac{\log\xi}{\xi}\right)_+ f(\xi) = \int_0^1 d\xi\,\log\xi\,\frac{f(\xi)-f(0)}{\xi}\,, \tag{2.51}$$

$$\int_{-1}^1 dy\,\left(\frac{1}{1-y}\right)_+ f(y) = \int_{-1}^1 dy\,\frac{f(y)-f(1)}{1-y}\,. \tag{2.52}$$

Inserting eqs. (2.48) and (2.49) into (2.47), and defining $g(\xi,y) \equiv \xi^2\,(1-y)\,\mathcal{R}$, for ease of notation, we have

$$\begin{aligned}\int_{-1}^{1} dy\,(1-y)^{-1-\epsilon}\int_0^1 d\xi\,\xi^{-1-2\epsilon}\,g(\xi,y) = &-\frac{1}{2\epsilon}\int_{-1}^1 dy\,(1-y)^{-1-\epsilon}g(0,y)\\ &-\int_0^1 d\xi\left[\frac{2^{-\epsilon}}{\epsilon}\left(\frac{1}{\xi}\right)_+ - 2\left(\frac{\log\xi}{\xi}\right)_+\right]g(\xi,1)\\ &+\int_{-1}^1 dy\int_0^1 d\xi\left(\frac{1}{\xi}\right)_+\left(\frac{1}{1-y}\right)_+ g(\xi,y) + \mathcal{O}(\epsilon)\,.\end{aligned} \tag{2.53}$$

The first term on the r.h.s. of eq. (2.53), after integration in y and ϕ, gives a contribution with the same structure as the virtual term, with which it is combined. Since this term arises from the $\delta(\xi)$ factor, it can be easily obtained using the eikonal approximation for soft emissions in d dimensions. The second term on the r.h.s. of eq. (2.53) is the contribution to $\mathcal{R}$ proportional to the delta-function term in eq. (2.49), multiplied by the second and third term of eq. (2.48). It gives rise to terms that, in the case of final-state singularities, can also be integrated in ξ and in ϕ, and yields terms of the same form of the virtual terms, with which it is combined. Also for this term it is not necessary to know $\mathcal{R}$ in d dimensions, since one can use the collinear approximation in the $y \to 1$ limit in order to obtain it. In the case of initial-state singularities, the same procedure gives terms of the same form of the collinear counterterms, with which they are combined. Finally, a term of the form

$$\int_{-1}^{1} dy \int_0^1 d\xi \left(\frac{1}{\xi}\right)_+ \left(\frac{1}{1-y}\right)_+ g(\xi, y) = \int_{-1}^{1} dy \int_0^1 d\xi\, \xi\, \hat{\mathcal{R}} \tag{2.54}$$

remains, where

$$\hat{\mathcal{R}} \equiv \frac{1}{\xi} \left\{ \left(\frac{1}{\xi}\right)_+ \left(\frac{1}{1-y}\right)_+ \left[\xi^2 (1-y)\, \mathcal{R}\right] \right\} . \tag{2.55}$$

Observe that the $1/\xi$ factor in front of eq. (2.55) cancels against the phase-space ξ factor in eq. (2.54), and that $[\xi^2(1-y)\,\mathcal{R}]$ has no singularities at $\xi = 0$ and $y = 1$, so that distributions in ξ and y act on a regular function.

The procedure outlined in this section is fully general. It can be shown that, defining $\hat{R} = \mathcal{L}\,\hat{\mathcal{R}}$, one can rewrite eq. (2.42) in the form

$$\begin{aligned} \langle O \rangle \; = & \int d\mathbf{\Phi}_n\, O_n(\mathbf{\Phi}_n) \left[B(\mathbf{\Phi}_n) + V(\mathbf{\Phi}_n)\right] + \int d\mathbf{\Phi}_{n+1}\, O_{n+1}(\mathbf{\Phi}_{n+1})\; \hat{R}(\mathbf{\Phi}_{n+1}) \\ & + \int d\mathbf{\Phi}_{n,\oplus}\, O_n\big(\bar{\mathbf{\Phi}}_n\big)\; G_\oplus(\mathbf{\Phi}_{n,\oplus}) + \int d\mathbf{\Phi}_{n,\ominus}\, O_n\big(\bar{\mathbf{\Phi}}_n\big)\; G_\ominus(\mathbf{\Phi}_{n,\ominus}) \ . \end{aligned} \tag{2.56}$$

By handling the + distributions in $\hat{\mathcal{R}}$ according to the prescriptions (2.50), (2.51) and (2.52), one automatically generates the real counterterms, provided the variables ξ and y appear in the phase-space parametrization. If more than one singular region is present, the real cross section is decomposed into a sum of terms, each of them having singularities in no more than one singular region. For each term, the phase space is parametrized in such a way that the variables ξ and y, appropriate to that particular singular region, are present.

The expression of a cross section in terms of distributions has sometimes the advantage that the associated projections are not uniquely fixed. In fact, one has the freedom to chose the integration variables other than y and ξ at one's convenience. This amounts to choosing a different projection.

2.4 Frixione, Kunszt and Signer subtraction

In this section, we briefly review the FKS general subtraction formalism, proposed in refs. [26, 27], including a few modifications that have been introduced recently (see ref. [14]).

In FKS one expresses the cross section for the real-emission contribution as a sum of terms, each of them having at most one collinear and one soft singularity associated with one parton (called the FKS parton). The singular region associated with the final-state parton i becoming soft or collinear to one of the incoming partons are labeled by i, while those associated with final-state parton i becoming soft or collinear to a final-state parton j are labeled by the pair ij. For each singular region one introduces certain non-negative functions[9] $\mathcal{S}$ of the $(n+1)$-body phase space such that

$$\sum_i \mathcal{S}_i + \sum_{ij} \mathcal{S}_{ij} = 1\,. \tag{2.57}$$

[9]The notation of refs. [26, 27] has been slightly changed here in order to simplify the discussion. Functions $\mathcal{S}_i$ and $\mathcal{S}_{ij}$ of the present paper play the same role as $\Theta_i^{(0)}$ and $\Theta_{ij}^{(1)}\theta(k_{jT}^2 - k_{iT}^2)$ of ref. [27] respectively.

We have two options for the range of the indices in the sums: we can let them range from 1 to $(n+1)$ (excluding only the $i = j$ possibility in the second sum), or we can assume that the $\mathcal{S}_i$ and $\mathcal{S}_{ij}$ are zero (i.e. they are excluded from the sum) if the corresponding regions are not singular. For example, if ij refer to a quark and an antiquark of different flavour there cannot be a FSC singularity in this region, and we can set $\mathcal{S}_{ij} = 0$. Also, if i is a gluon and j is a quark, we may set to zero $\mathcal{S}_{ji}$, since there is no soft singularity associated to j becoming soft. Notice that if i and j are both gluons, both terms $\mathcal{S}_{ij}$ and $\mathcal{S}_{ji}$ appear in the sum, since there is a soft singularity for either parton becoming soft.

The $\mathcal{S}$ function have the following properties

$$\lim_{k_m^0 \to 0} \left(\mathcal{S}_i + \sum_j \mathcal{S}_{ij} \right) = \delta_{im} \,, \tag{2.58}$$

$$\lim_{\vec{k}_m \| \vec{k}_{\oplus}} \mathcal{S}_i = \delta_{im} \,, \tag{2.59}$$

$$\lim_{\vec{k}_m \| \vec{k}_l} (\mathcal{S}_{ij} + \mathcal{S}_{ji}) = \delta_{im}\delta_{jl} + \delta_{il}\delta_{jm} \,, \tag{2.60}$$

$$\lim_{\vec{k}_m \| \vec{k}_{\oplus}} \mathcal{S}_{ij} = 0 \,. \tag{2.61}$$

Thus, in a given soft region, (i.e. if parton m is soft), all $\mathcal{S}_i$ and $\mathcal{S}_{ij}$ with $i \neq m$ vanish, and eq. (2.57) is consistent with eq. (2.58). In a given initial-state collinear region, i.e. parton m becomes collinear to an initial state parton, only $\mathcal{S}_m$ is non-zero and equal to one, consistently with eq. (2.57). In a given final-state collinear region, i.e. if partons i and j are collinear, only $\mathcal{S}_{ij}$ and $\mathcal{S}_{ji}$ can differ from zero, their sum being 1, again consistently with eq. (2.57).

We now write

$$\mathcal{R} = \sum_i \mathcal{R}_i + \sum_{ij} \mathcal{R}_{ij} \,, \tag{2.62}$$

where

$$\mathcal{R}_i = \mathcal{S}_i \mathcal{R} \,, \qquad \mathcal{R}_{ij} = \mathcal{S}_{ij} \mathcal{R} \,. \tag{2.63}$$

The $\mathcal{R}_i$ terms give a divergent contribution (i.e. a contribution which has to be subtracted) only in the regions in which parton i is soft and/or collinear to one of the initial-state partons, and the $\mathcal{R}_{ij}$ terms are divergent only in the regions in which parton i is soft and/or collinear to final-state parton j. Notice that if we have chosen the option of keeping all the $\mathcal{S}$ functions different from zero, the $\mathcal{R}_i$ and $\mathcal{R}_{ij}$ functions corresponding to non-singular regions are non-zero but finite.

Equations (2.58)–(2.61) are the only properties of the $\mathcal{S}$ functions used in the analytical computations of refs. [26, 27]. Their actual functional forms, away from the infrared limits, are only relevant to numerical integrations.

After the $(n + 1)$-body cross section is decomposed as in eq. (2.62), FKS chooses a different parametrization of the $(n + 1)$-body phase space for each term, such that one can perform the necessary analytical and numerical integrations in an easy way. The key variables in the phase-space parametrization associated with $\mathcal{S}_i$ are the energy of parton

i (directly related to soft singularities), and the angle between parton i and one of the initial-state partons (directly related to initial-state collinear singularities). For $\mathcal{S}_{ij}$, the energy of parton i and the angle between parton i and j (related to a final-state collinear singularity) are chosen instead. Therefore, there are only two independent functional forms for phase spaces in FKS, one for initial- and one for final-state emissions.

In the following, we will need a further refinement of the FKS decomposition (eq. (2.57)). This is because, in FKS, the $\oplus$ and $\ominus$ collinear regions are both singled out by the $\mathcal{S}_i$ functions. In POWHEG we will need sometimes to treat the two collinear regions separately. We thus introduce the notation

$$\mathcal{S}_i = \mathcal{S}_i^{\oplus} + \mathcal{S}_i^{\ominus} \,, \tag{2.64}$$

with the properties

$$\lim_{\vec{k}_m \parallel \vec{k}_{\oplus/\ominus}} \mathcal{S}_i^{\oplus/\ominus} = \delta_{im} \,, \qquad \lim_{\vec{k}_m \parallel \vec{k}_{\oplus/\ominus}} \mathcal{S}_i^{\ominus/\oplus} = 0 \,, \tag{2.65}$$

that refine eq. (2.59). Eqs. (2.62) and (2.63) are modified accordingly.

2.4.1 The $\mathcal{S}$ functions

In the original formulation of the FKS subtraction, the $\mathcal{S}$ functions were defined as sets of θ functions. The different contributions to the real cross section, separated out in this way, corresponded to a partition of the phase space into non-overlapping regions. In the more recent calculation of single-top hadroproduction in `MC@NLO` [14], the $\mathcal{S}$ functions were instead defined as smooth functions. In view of Monte Carlo implementations, step functions should be avoided as much as possible, and therefore we consider here the latter approach. We introduce the functions d_i and d_{ij}, where $i, j = 1, \ldots, n+1$, with the following properties

$$\begin{aligned} d_i = 0 \quad & \text{if and only if} \quad E_i = 0 \;\; \text{or} \;\; \vec{k}_i \parallel \vec{k}_{\oplus} \;\; \text{or} \;\; \vec{k}_i \parallel \vec{k}_{\ominus} \,, \\ d_{ij} = 0 \quad & \text{if and only if} \quad E_i = 0 \;\; \text{or} \;\; E_j = 0 \;\; \text{or} \;\; \vec{k}_i \parallel \vec{k}_j \,, \end{aligned} \tag{2.66}$$

where energies and three-momenta are computed in the centre-of-mass frame of the incoming partons. A possible definition of the d's is

$$d_i = \left(\frac{\sqrt{s}}{2} E_i \right)^a \left(1 - \cos^2 \theta_i\right)^b \,, \tag{2.67}$$

$$d_{ij} = \left(E_i E_j \right)^a \left(1 - \cos\theta_{ij}\right)^b \,, \tag{2.68}$$

where θ_i is the angle between $\vec{k}_i$ and $\vec{k}_{\oplus}$, θ_{ij} the angle between $\vec{k}_i$ and $\vec{k}_j$, $s = (k_{\oplus} + k_{\ominus})^2$, and a, b are positive real numbers. Equations (2.67) and (2.68) can be easily expressed in terms of invariants using

$$k_{\oplus} = \frac{\sqrt{s}}{2}(1,0,0,1) \,, \quad k_{\ominus} = \frac{\sqrt{s}}{2}(1,0,0,-1) \,, \tag{2.69}$$

which imply

$$E_i = \frac{1}{\sqrt{s}} \, (k_{\oplus} + k_{\ominus}) \cdot k_i \,, \tag{2.70}$$

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$$\cos\theta_i = 1 - \frac{2k_i\cdot k_\oplus}{E_i\sqrt{s}}\,, \tag{2.71}$$

$$\cos\theta_{ij} = 1 - \frac{k_i\cdot k_j}{E_iE_j}\,. \tag{2.72}$$

We now introduce the quantity

$$\mathcal{D} = \sum_k \frac{1}{d_k} + \sum_{kl}\frac{1}{d_{kl}}\,, \tag{2.73}$$

and define

$$\mathcal{S}_i = \frac{1}{\mathcal{D}\,d_i}\,, \tag{2.74}$$

$$\mathcal{S}_{ij} = \frac{1}{\mathcal{D}\,d_{ij}}\,h\!\left(\frac{E_i}{E_i+E_j}\right)\,, \tag{2.75}$$

where h is a function such that

$$\lim_{z\to 0} h(z) = 1\,, \qquad \lim_{z\to 1} h(z) = 0\,, \qquad h(z) + h(1-z) = 1\,. \tag{2.76}$$

For example, one can define[10]

$$h(z) = \frac{(1-z)^c}{z^c + (1-z)^c} \tag{2.77}$$

for some positive c. Notice that the h factor is necessary only if one considers both functions $\mathcal{S}_{ij}$ and $\mathcal{S}_{ji}$ (which is strictly necessary only if both i and j are gluons).

It is manifest that eqs. (2.74) and (2.75) fulfill eqs. (2.58)–(2.61).

As in section 2.4, in order to separate the $\oplus$ and $\ominus$ collinear regions, we modify the previous formulae as follows. We introduce

$$d_i^{\substack{\oplus\\ \ominus}} = \left(\frac{\sqrt{s}}{2}E_i\right)^a 2^b(1\mp\cos\theta_i)^b\,, \tag{2.78}$$

instead of d_i of eq. (2.67). The definition of $\mathcal{D}$ in eq. (2.73) becomes

$$\mathcal{D} = \sum_k \left(\frac{1}{d_k^\oplus} + \frac{1}{d_k^\ominus}\right) + \sum_{kl}\frac{1}{d_{kl}}\,. \tag{2.79}$$

We then define

$$\mathcal{S}_i^{\substack{\oplus\\ \ominus}} = \frac{1}{\mathcal{D}\,d_i^{\substack{\oplus\\ \ominus}}}\,. \tag{2.80}$$

[10] In the original FKS formulation $h(z) = \theta(1/2 - z)$, which implies that $\mathcal{S}_{ij}$ defined in eq. (2.75) vanishes if $E_j < E_i$. With such a choice, a proof was given in ref. [26] that all the infrared singularities are canceled by the FKS subtraction. We have checked that, if a smooth form for h is adopted, this proof goes through unaltered by replacing $\theta(z - 1/2)$ in eq. (4.84) and (4.85) of ref. [26] with $h(1-z)$.

2.4.2 Contributions to the cross section

We introduce, for the FKS parton i, the following variables

$$\xi_i = \frac{2k_i^0}{\sqrt{s}}\,, \qquad y_i = \cos\theta_i\,, \qquad y_{ij} = \cos\theta_{ij}\,, \tag{2.81}$$

where θ_i is the angle of parton i with the incoming parton $\oplus$, and θ_{ij} is the angle of parton i with parton j. All variables are computed in the centre-of-mass frame of the incoming partons. The phase space for the $\mathcal{R}_i$ and $\mathcal{R}_{ij}$ contributions is written in $d = 4 - 2\epsilon$ dimensions as

$$\begin{aligned} d\Phi_{n+1} &= (2\pi)^d\, \delta^d\!\left(k_\oplus + k_\ominus - \sum_{i=1}^{n+1} k_i\right) \left[\prod_{l\neq i} \frac{d^{d-1}k_l}{(2\pi)^{d-1}\, 2k_l^0}\right] \\ &\quad\times \frac{s^{1-\epsilon}}{(4\pi)^{d-1}}\, \xi_i^{1-2\epsilon} \left(1-y^2\right)^{-\epsilon} d\xi_i\, dy\, d\Omega^{d-2}, \end{aligned} \tag{2.82}$$

where y, Ω stands for either y_i, Ω_i or y_{ij}, Ω_{ij}. The transverse angular variables $d\Omega_i^{d-2}$ are relative to the collision axis, while $d\Omega_{ij}^{d-2}$ are relative to the direction of parton j. The singularities for $\xi \to 0$, $y_i \to \pm 1$ or $y_{ij} \to 1$ are treated along the lines of section 2.3. The final expression in the FKS formalism results from the cancellation of the infrared singularities which emerge in the intermediate steps of the computation. It thus involves only non-divergent terms.

The contributions to the real-emission cross section, in the notation of eq. (2.56), are

$$\hat{\mathcal{R}} = \sum_i \hat{\mathcal{R}}_i + \sum_{ij} \hat{\mathcal{R}}_{ij}\,, \tag{2.83}$$

where

$$\hat{\mathcal{R}}_i = \frac{1}{\xi_i}\left\{ \frac{1}{2}\left(\frac{1}{\xi_i}\right)_{\xi_c} \left[\left(\frac{1}{1-y_i}\right)_{\delta_\mathrm{I}} + \left(\frac{1}{1+y_i}\right)_{\delta_\mathrm{I}}\right] \left[\left(1-y_i^2\right)\, \xi_i^2\, \mathcal{R}_i\right] \right\}, \tag{2.84}$$

$$\hat{\mathcal{R}}_{ij} = \frac{1}{\xi_i}\left\{ \left(\frac{1}{\xi_i}\right)_{\xi_c} \left(\frac{1}{1-y_{ij}}\right)_{\delta_\mathrm{O}} \left[(1-y_{ij})\, \xi_i^2\, \mathcal{R}_{ij}\right] \right\}. \tag{2.85}$$

If we need to separate the $\oplus$ and $\ominus$ collinear regions, as discussed at the end of section 2.4, we have

$$\hat{\mathcal{R}} = \sum_i \left(\hat{\mathcal{R}}_i^\oplus + \hat{\mathcal{R}}_i^\ominus\right) + \sum_{ij} \hat{\mathcal{R}}_{ij}\,, \tag{2.86}$$

$$\hat{\mathcal{R}}_i^{\oplus\atop\ominus} = \frac{1}{\xi_i}\left\{ \left(\frac{1}{\xi_i}\right)_{\xi_c} \left(\frac{1}{1\mp y_i}\right)_{\delta_\mathrm{I}} \left[(1\mp y_i)\, \xi_i^2\, \mathcal{R}_i^{\oplus\atop\ominus}\right] \right\}. \tag{2.87}$$

The distributions that appear in eqs. (2.84) and (2.85) are defined as follows

$$\int_0^1 d\xi\, f(\xi) \left(\frac{1}{\xi}\right)_{\xi_c} = \int_0^1 d\xi\, \frac{f(\xi) - f(0)\,\theta(\xi_c - \xi)}{\xi}\,, \tag{2.88}$$

$$\int_{-1}^{1} dy\, f(y) \left(\frac{1}{1\mp y}\right)_{\delta} = \int_{-1}^{1} dy\, \frac{f(y) - f(\pm 1)\,\theta(\pm y - 1 + \delta)}{1 \mp y}. \tag{2.89}$$

The parameters ξ_c, $\delta_{\rm I}$ and $\delta_{\rm O}$ must be chosen in the ranges $0 < \xi_c \leq 1$ and $0 < \delta_{I,O} \leq 2$. The dependence they introduce in the $(n+1)$-body contribution is exactly compensated by the same dependence in the n-body contribution. In the POWHEG framework it is often convenient to use the maximal range of integration, and we will thus also use the notation

$$\left(\frac{1}{\xi}\right)_{+} = \left(\frac{1}{\xi}\right)_{\xi_c} \text{ with } \xi_c = 1\,, \qquad \left(\frac{1}{1\mp y}\right)_{+} = \left(\frac{1}{1\mp y}\right)_{\delta} \text{ with } \delta = 2\,. \tag{2.90}$$

The parametrization of the $(n+1)$-body phase space, appropriate to the integration of $\mathcal{R}_i$ and $\mathcal{R}_{ij}$, can be chosen as the $d = 4$ version of equation (2.82), as suggested in the original FKS papers. This is however not necessary. Any parametrization of the phase space that allows a simple handling of the distributions is acceptable. This freedom is exploited in the present work, in order to simplify the formulation of the POWHEG method in the case of the $\mathcal{R}_{ij}$ contributions, where we make a choice of the azimuthal variables different from that of eq. (2.82).

We now consider the soft-virtual term V in eq. (2.56). We define the set of all the $n+2$ parton labels for an n-body process

$$\mathcal{I} = \{\oplus, \ominus, 1, \ldots, n\}\,. \tag{2.91}$$

The virtual contribution $\mathcal{V}_{\rm b}$ of eq. (2.37) is given by[11]

$$\mathcal{V}_{\rm b} = \mathcal{N}\,\frac{\alpha_{\rm S}}{2\pi}\left[-\sum_{i\in\mathcal{I}}\left(\frac{1}{\epsilon^2}\,C_{f_i} + \frac{1}{\epsilon}\,\gamma_{f_i}\right)\mathcal{B} + \frac{1}{\epsilon}\sum_{\substack{i,j\in\mathcal{I}\\ i\neq j}} \log\frac{2k_i\cdot k_j}{Q^2}\,\mathcal{B}_{ij} + \mathcal{V}_{\rm fin}\right], \tag{2.92}$$

where

$$\mathcal{N} = \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}. \tag{2.93}$$

Notice that, in the second sum on the r.h.s. of eq. (2.92), we sum over $i \neq j$, and thus, since (as we show later) $\mathcal{B}_{ij}$ is symmetric, each term appears twice in the sum. The definition of the finite part $\mathcal{V}_{\rm fin}$ depends upon the definition of the normalization factor $\mathcal{N}$, for which we have adopted the common choice of eq. (2.93), and from the regularization scheme, that we assume to be the standard conventional dimensional regularization (CDR).[12] Finally, μ^2 is the renormalization scale, and Q^2 is an (arbitrary) physical scale that is factored out from the virtual amplitude in order to make the normalization $\mathcal{N}$ dimensionless (thus $\mathcal{V}_{\rm fin}$ depends upon μ^2 and Q^2).

The symbol f_i denotes the flavour of parton i, i.e. g for a gluon, q for a quark and $\bar{q}$ for an antiquark. We define

$$C_g = C_{\rm A}\,, \qquad\qquad C_q = C_{\bar{q}} = C_{\rm F}\,, \tag{2.94}$$

[11] We stress that $\mathcal{V}_{\rm b}$ is the contribution to the cross section due to the interference of the virtual amplitude with the Born term. It thus includes the corresponding factor of 2.

[12] If we instead use the dimensional reduction (DR) scheme, we have $\mathcal{V}_{\rm fin} = \mathcal{V}_{\rm fin}^{\rm DR} - \alpha_{\rm S}/(2\pi)\,\mathcal{B}\sum_{i\in\mathcal{I}}\tilde{\gamma}(f_i)$, where, $\tilde{\gamma}(g) = N_{\rm c}/6$ and $\tilde{\gamma}(q) = (N_{\rm c}^2 - 1)/(4N_{\rm c})$, where $N_{\rm c} = 3$ is the number of colours.

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$$\gamma_g = \frac{11C_{\rm A} - 4T_{\rm F}\, n_{\rm f}}{6}\,, \qquad\qquad \gamma_q = \gamma_{\bar q} = \frac{3}{2} C_{\rm F}\,, \tag{2.95}$$

$$\gamma'_g = \left(\frac{67}{9} - \frac{2\pi^2}{3}\right) C_{\rm A} - \frac{23}{9} T_{\rm F}\, n_{\rm f}\,, \qquad \gamma'_q = \gamma'_{\bar q} = \left(\frac{13}{2} - \frac{2\pi^2}{3}\right) C_{\rm F}\,. \tag{2.96}$$

In case i is a colorless particle all the above quantities are zero.

The quantities $\mathcal{B}_{ij}$, commonly referred to as the colour-correlated Born amplitudes, are defined in the following way

$$\mathcal{B}_{ij} = -\frac{1}{2s} \frac{1}{N_{\rm sym} D_\oplus\, D_\ominus\, S_\oplus\, S_\ominus} \sum_{\substack{\text{spins}\\ \text{colours}}} \mathcal{M}_{\{c_k\}} \left(\mathcal{M}^\dagger_{\{c_k\}}\right)_{\substack{c_i \to c'_i \\ c_j \to c'_j}} T^a_{c_i, c'_i}\, T^a_{c_j, c'_j}\,. \tag{2.97}$$

Here $\mathcal{M}_{\{c_k\}}$ is the Born amplitude, and $\{c_k\}$ stands for the colour indices of all partons in $\mathcal{I}$. The suffix on the parentheses that enclose $\mathcal{M}^\dagger_{\{c_k\}}$ indicates that the colour indices of partons $i, j \in \mathcal{I}$ are substituted with primed indices in $\mathcal{M}^\dagger_{\{c_k\}}$. Furthermore $N_{\rm sym}$ is the symmetry factor for identical particles in the final state, $D_{\oplus\!\!\!\ominus}$ are the dimension of the colour representations of the incoming partons (3 for quarks and 8 for gluons), and $S_{\oplus\!\!\!\ominus}$ are the number of spin states. The factor $1/(2s)$ is the flux factor. We assume summation over repeated colour indexes (c_k for $k \in \mathcal{I}$, c'_i, c'_j and a) and spin indices. For gluons $T^a_{cb} = i f_{cab}$, where f_{abc} are the structure constants of the $SU(3)$ algebra. For incoming quarks $T^a_{\alpha\beta} = t^a_{\alpha\beta}$, where t are the colour matrices in the fundamental representation (normalized as ${\rm Tr}[t\,t] = 1/2$). For antiquarks $T^a_{\alpha\beta} = -t^a_{\beta\alpha}$. It follows from colour conservation that $\mathcal{B}_{ij}$ satisfy

$$\sum_{i \in \mathcal{I}, i \neq j} \mathcal{B}_{ij} = C_{f_j} \mathcal{B}\,. \tag{2.98}$$

The soft-virtual term in eq. (2.56) is given by

$$V = \mathcal{L}\, \mathcal{V}\,, \qquad \mathcal{V} = \frac{\alpha_{\rm S}}{2\pi} \left(\mathcal{Q}\, \mathcal{B} + \sum_{\substack{i,j \in \mathcal{I} \\ i \neq j}} \mathcal{I}_{ij}\, \mathcal{B}_{ij} + \mathcal{V}_{\rm fin} \right). \tag{2.99}$$

The quantities $\mathcal{Q}$ and $\mathcal{I}_{ij}$ depend on the flavours and momenta of the incoming and outgoing partons. They are defined as follows

$$\begin{aligned} \mathcal{Q} = \sum_{i=1}^{n} \Bigg[& \gamma'_{f_i} - \log\frac{s\delta_{\rm O}}{2Q^2} \left(\gamma_{f_i} - 2C_{f_i} \log\frac{2E_i}{\xi_c \sqrt{s}} \right) \\ & + 2C_{f_i} \left(\log^2 \frac{2E_i}{\sqrt{s}} - \log^2 \xi_c \right) - 2\gamma_{f_i} \log\frac{2E_i}{\sqrt{s}} \Bigg] \\ & - \log\frac{\mu_{\rm F}^2}{Q^2} \left[\gamma_{f_\oplus} + 2C_{f_\oplus} \log \xi_c + \gamma_{f_\ominus} + 2C_{f_\ominus} \log \xi_c \right]\,, \end{aligned} \tag{2.100}$$

$$\begin{aligned} \mathcal{I}_{ij} = & \frac{1}{2} \log^2 \frac{\xi_c^2 s}{Q^2} + \log\frac{\xi_c^2 s}{Q^2} \log\frac{k_j \cdot k_i}{2E_j E_i} - {\rm Li}_2\left(\frac{k_j \cdot k_i}{2E_j E_i}\right) \\ & + \frac{1}{2} \log^2 \frac{k_j \cdot k_i}{2E_j E_i} - \log\left(1 - \frac{k_j \cdot k_i}{2E_j E_i}\right) \log\frac{k_j \cdot k_i}{2E_j E_i}\,, \end{aligned} \tag{2.101}$$

where E_i is the energy of parton i in the partonic centre-of-mass frame.

We finally report the expressions for the initial-state collinear remnants that appear in eq. (2.56). For each collinear singular configuration, relevant to the process, we have a term $G_\oplus = \tilde{\mathcal{L}}\,\mathcal{G}_\oplus$ (and a corresponding one for $G_\ominus = \tilde{\mathcal{L}}\,\mathcal{G}_\ominus$), where

$$\mathcal{G}_\oplus^{f_\oplus f_\ominus}(z) = \frac{\alpha_{\rm S}}{2\pi}\sum_{f'_\oplus}\left\{(1-z)\,P^{f_\oplus f'_\oplus}(z,0)\left[\left(\frac{1}{1-z}\right)_{\xi_c}\log\frac{s\delta_{\rm I}}{2\mu_{\rm F}^2}+2\left(\frac{\log(1-z)}{1-z}\right)_{\xi_c}\right]\right.$$
$$\left. -\left[\frac{\partial P^{f_\oplus f'_\oplus}(z,\epsilon)}{\partial\epsilon}\right]_{\epsilon=0} - K^{f_\oplus f'_\oplus}(z)\right\}\mathcal{B}^{f'_\oplus f_\ominus}(z)\,, \tag{2.102}$$

for a process in which an incoming parton $\oplus$ of flavour $f_\oplus$ splits into a parton $f'_\oplus$ (with fraction z of the incoming momentum) that enters the Born process $\mathcal{B}$. The superscripts on $\mathcal{B}$ denote the flavours of the incoming parton, and the z dependence is to remind that the incoming $\oplus$ momentum is rescaled. The distributions are defined as in eq. (2.88), with $\xi = 1-z$. The functions $P(z,\epsilon)$ are the leading order Altarelli-Parisi splitting functions in $d = 4-2\epsilon$ dimensions, given by

$$P^{qq}(z,\epsilon) = C_{\rm F}\left[\frac{1+z^2}{1-z}-\epsilon(1-z)\right]\,, \tag{2.103}$$

$$P^{qg}(z,\epsilon) = C_{\rm F}\left[\frac{1+(1-z)^2}{z}-\epsilon z\right]\,, \tag{2.104}$$

$$P^{gq}(z,\epsilon) = T_{\rm F}\left[1-\frac{2z(1-z)}{1-\epsilon}\right]\,, \tag{2.105}$$

$$P^{gg}(z,\epsilon) = 2\,C_{\rm A}\left[\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\right]\,. \tag{2.106}$$

The distributions $K^{ff'}$ control the change of scheme in the evolution of parton distribution functions. They are defined in ref. [26], and equivalently, with the notation $K^{ff'}_{\rm F.S.}$ in ref. [28]. They are identically zero in $\overline{\rm MS}$.

2.5 Catani and Seymour subtraction

In this section we briefly review the general subtraction formalism proposed in ref. [28], called *dipole subtraction*. A dipole is defined by three partons: the emitted, the emitter and the spectator parton (the last two forming the dipole). In the dipole formulation, one given singular region receives, in general, contributions by several dipoles, differing among each other by the spectator parton. Thus, the counterterms $\mathcal{C}^{(\alpha)}$ are associated with dipoles, rather than singular regions. The maps $\mathbf{M}^{(\alpha)}$ of eq. (2.18) in the dipole formulation (that are summarized in section 6) are constructed in such a way that, in most cases, they affect only the momenta of the dipole partons, and all other momenta remain unchanged.[13] The maps $\mathbf{M}^{(\alpha)}$ appropriate to the dipole subtraction will be discussed in section 6. These definitions, together with the definitions of the relative dipole counterterms (to be found in the original reference [28]) are necessary to define a POWHEG implementation. The

[13]The only exception is when the emitter and the spectator are the two incoming partons.

last two missing ingredients are the soft-virtual term $\mathcal{V}$, and the collinear remnants $\mathcal{G}_{\oplus}$. In this section we report explicitly the form of these terms, expressed in our notation.

For the soft-virtual contribution $\mathcal{V}$ we obtain the following

$$\mathcal{V} = \frac{\alpha_{\rm S}}{2\pi}\left\{\mathcal{V}_{\rm fin} - \sum_{\substack{i,j\in\mathcal{I}\\ i\neq j}}\left[\frac{1}{2}\log^2\frac{Q^2}{2k_i\cdot k_j} + \frac{\gamma_{f_i}}{C_{f_i}}\log\frac{Q^2}{2k_i\cdot k_j}\right]\mathcal{B}_{ij} + \sum_{i\in\mathcal{I}}\left[-\frac{\pi^2}{3}C_{f_i} + \gamma_{f_i} + K_{f_i}\right]\mathcal{B}\right\}. \tag{2.107}$$

Equation (2.107) has been obtained by a suitable manipulation of eqs. (C.27) and (C.28) of ref. [28]. The virtual term $\mathcal{V}_{\rm fin}$ coincides with that of our eq. (2.92).

The collinear remnant is given by

$$\begin{aligned}\mathcal{G}_{\oplus}^{f_\oplus f_\ominus}(z) = \frac{\alpha_{\rm S}}{2\pi}\sum_{f'_\oplus}\Bigg\{&\left[\overline{K}^{f_\oplus f'_\oplus}(z) - K_{\rm F.S.}^{f_\oplus f'_\oplus}(z)\right]\mathcal{B}^{f'_\oplus f_\ominus}(z)\\ &-\delta^{f_\oplus f'_\oplus}\sum_{i=1}^{n}\frac{\gamma_{f_i}}{C_{f_i}}\left[\left(\frac{1}{1-z}\right)_+ + \delta(1-z)\right]\mathcal{B}_{i\oplus}^{f'_\oplus f_\ominus}(z) + \frac{1}{C_{f'_\oplus}}\tilde{K}^{f_\oplus f'_\oplus}(z)\,\mathcal{B}_{\ominus\oplus}^{f'_\oplus f_\ominus}(z)\\ &-P^{f_\oplus f'_\oplus}(z)\frac{1}{C_{f'_\oplus}}\sum_{\substack{i\in\mathcal{I}\\ i\neq\oplus}}\mathcal{B}_{i\oplus}^{f'_\oplus f_\ominus}(z)\log\frac{\mu_{\rm F}^2}{2\,z\,k_\oplus\cdot k_i}\Bigg\},\end{aligned} \tag{2.108}$$

and the analogous one for $\mathcal{G}_\ominus$. This formula is the translation in our notation of eq. (10.30) of ref. [28]. The definition of the functions $\overline{K}$, $\tilde{K}$, $K_{\rm F.S.}$ and P are given in appendix C of ref. [28].

In eq. (2.108), $\mathcal{G}_{\oplus}^{f_\oplus f_\ominus}$ is the collinear remnant contribution for flavours $f_\oplus, f_\ominus$ of the incoming partons. Analogously, the superscripts in $\mathcal{B}$ (and $\mathcal{B}_{ij}$) single out a given flavour combination for the incoming partons in the Born amplitudes and its colour-correlated components.

3. NLO with parton showers

3.1 Parton shower Monte Carlo programs

A detailed discussion of the ideas upon which Shower Monte Carlo programs are based is beyond the scope of the present paper. The interested reader can find some pedagogical introductions e.g. in refs. [30, 31]. Here, we simply need to recall few basic features of SMC programs.

First, an MC starts from a kinematic configuration ("hard") which is generated according to an exact LO computation. Usually such configuration is that of a $2\to 2$ partonic process. The final-state multiplicity is then iteratively increased, by letting each initial- and final-state parton branch into a couple of partons with a probability related to a Sudakov form factor. Thus, if at a given stage of the shower, the scattering process is described by m partons, the algorithm decides with a certain probability whether branching is over at this stage, or further branchings will take place. In the latter case, one of the m partons splits into a pair, generating an $m+1$ body final state. Thus, the algorithm defines a mapping

$$\mathbf{\Phi}_{m+1} \overset{(\alpha)}{\Longleftrightarrow} \left\{ \bar{\mathbf{\Phi}}_m^{(\alpha)}, \Phi_{\rm rad}^{(\alpha)} \right\}, \tag{3.1}$$

that is fully analogous to the mapping (2.28). Also in this case there is one mapping for each singular region, where the singular region is associated with the parton that undergoes the splitting. Observe that mappings defined in eq. (3.1) act non-trivially also on the momenta of the partons that do not undergo any splitting process. This is due to the fact that momentum conservation must be restored after branching, an operation that is usually referred to as "momentum reshuffling" in the SMC jargon. Also the value of the momentum fraction of the incoming partons may require readjustment, which leads to the fact that the value of the luminosity used for the cross-section computation does not correspond exactly to what one would have used if the $(m+1)$-particle matrix element had been computed with standard methods. These readjustments usually amount to corrections beyond the (leading log) Monte Carlo accuracy, but should be analyzed carefully if one aims at NLO accuracy.

3.2 Including NLO corrections into Monte Carlo programs

The embedding of a NLO computation into an MC framework, as first clarified in ref. [2], aims at reaching NLO accuracy for inclusive observable, maintaining the leading logarithmic accuracy of the shower approach. This requires that the hardest emission that is generated has the correct distribution also far from the collinear directions, and that integrated quantities around the soft and collinear directions have NLO accuracy. This requirements are met in the `MC@NLO` approach by carefully tracing the differences of the MC simulation relative to the exact NLO one. The similarity of the mappings (3.1) and (2.28) are the starting point for this task. The shower algorithm is analyzed to determine its own approximate NLO structure in the subtraction framework, in order to determine unambiguously the difference with the exact NLO formulae.

In the POWHEG approach, one performs the generation of the hardest event with NLO accuracy, in a framework that does not depend upon the SMC's shower algorithm. This is why it is fully independent from the SMC. Furthermore, the subsequent showers takes place at softer transverse momenta, and thus affects infrared-safe observables only at the next-to-next-to-leading order (NNLO). Thus, the matching problem considerably simplifies, since it no longer requires a detailed examination of the properties of the SMC.

3.3 POWHEG

In the POWHEG formalism, the generation of the hardest emission is performed first, using full NLO accuracy, and using the SMC to generate subsequent radiation. We give here a simple illustration of the method, ignoring, for the moment, the complications due to the presence of several singular regions in the NLO cross section. We begin by defining

$$\begin{aligned}\bar{B}(\mathbf{\Phi}_n) &= B(\mathbf{\Phi}_n) + V(\mathbf{\Phi}_n) \\ &+ \left[\int d\Phi_{\rm rad} \left[R(\mathbf{\Phi}_{n+1}) - C(\mathbf{\Phi}_{n+1}) \right] + \int \frac{dz}{z} \left[G_\oplus(\mathbf{\Phi}_{n,\oplus}) + G_\ominus(\mathbf{\Phi}_{n,\ominus}) \right] \right]^{\bar{\mathbf{\Phi}}_n = \mathbf{\Phi}_n}, \end{aligned} \tag{3.2}$$

where we have assumed that all the $\mathbf{\Phi}_{n+1}$, $\mathbf{\Phi}_{n,\oplus}$ are expressed in terms of the barred variables. Next we introduce the Sudakov form factor[14]

$$\Delta\left(\mathbf{\Phi}_n, p_{\mathrm{T}}\right) = \exp\left\{ -\int \frac{\left[d\Phi_{\mathrm{rad}}\, R(\mathbf{\Phi}_{n+1})\, \theta(k_{\mathrm{T}}\left(\mathbf{\Phi}_{n+1}\right) - p_{\mathrm{T}})\right]^{\bar{\mathbf{\Phi}}_n = \mathbf{\Phi}_n}}{B(\mathbf{\Phi}_n)} \right\}. \tag{3.3}$$

The function $k_{\mathrm{T}}\left(\mathbf{\Phi}_{n+1}\right)$ should be equal, near the singular limit, to the transverse momentum of the emitted parton relative to the emitting one. The POWHEG cross section for the generation of the hardest event is then

$$d\sigma = \bar{B}(\mathbf{\Phi}_n)\, d\mathbf{\Phi}_n \left\{ \Delta\left(\mathbf{\Phi}_n, p_{\mathrm{T}}^{\mathrm{min}}\right) + \Delta\left(\mathbf{\Phi}_n, k_{\mathrm{T}}\left(\mathbf{\Phi}_{n+1}\right)\right) \frac{R\left(\mathbf{\Phi}_{n+1}\right)}{B(\mathbf{\Phi}_n)} d\Phi_{\mathrm{rad}} \right\}_{\bar{\mathbf{\Phi}}_n = \mathbf{\Phi}_n}, \tag{3.4}$$

where it is assumed that $\mathbf{\Phi}_{n+1}$ is parametrized in terms of Φ_{rad} and $\mathbf{\Phi}_n$, and that values of $k_{\mathrm{T}}\left(\mathbf{\Phi}_{n+1}\right) < p_{\mathrm{T}}^{\mathrm{min}}$ are not allowed. The cross section (3.4) has the following properties:

- At large k_{T} it coincides with the NLO cross section up to NNLO terms.
- It reproduces correctly the value of infrared safe observables at the NLO. Thus, also its integral around the small k_{T} region has NLO accuracy.
- At small k_{T} it behaves no worse than standard Shower Monte Carlo generators.

Thus, it fulfills the requirement of the previous subsection for the inclusion of NLO corrections in an SMC.

As it stands, the POWHEG formula (3.4) can be used to feed a SMC program, that will perform all subsequent (softer) showers and hadronization. If the SMC is ordered in p_{T}, we simply require that the shower is started with an upper limit on the scale equal to the k_{T} of the POWHEG event. In case the SMC uses a different ordering variable, a problem arises, since the POWHEG cross section requires the emissions with higher k_{T} to be suppressed in the SMC. This problem typically arises when interfacing POWHEG to angular ordered SMC's. It is dealt with by vetoing emissions with larger k_{T} in the shower, and by introducing vetoed truncated showers (see ref. [4]), that compensate for the fact that in angular ordered shower the hardest emission may not be the first.

Modern SMC programs, such as `HERWIG` and `PYTHIA`, have the capability of generating a vetoed shower. This is not the case for the vetoed truncated showers. We point out, however, that the need of vetoed truncated showers is not specific to the POWHEG method. As discussed in ref. [4], it also emerges naturally when interfacing standard matrix element calculations with parton shower, as in the approach of ref. [33]. At present, there is no evidence that the effect of vetoed truncated showers may have any practical important.

[14]Torbjörn Sjöstrand has pointed out to us that a similar Sudakov form factor is also used in `PYTHIA` for weak vector-bosons decay and production, in order to implement a matrix-element matching for the first emission in the shower, see refs. [31, 32].

4. The POWHEG method

In order to implement the POWHEG method one must specify the separation of the singular regions, and the kinematics that associates a given $(n+1)$-body singular region with an n-body one. We discuss POWHEG in the framework of a generic subtraction formalism.

When one aims at the construction of an event generator, flavour should be carefully tracked, since different flavour structures always give rise to different events. We thus distinguish the contributions to the cross section also by their flavour structures, which are determined by the flavours of the incoming and outgoing partons. We call equivalent two flavour structures that differ only by a permutation of final-state partons. In particular, we label with the index f_b the flavour structure of the n-body processes and write B^{f_b} and V^{f_b} for the various Born and soft-virtual contributions.

We label with $\alpha_{\rm r}$ a particular contribution to the real cross section that is singular in only one singular region of integration and has a specific flavour structure. Thus, to each $\alpha_{\rm r}$ corresponds one and only one singular region. We then write

$$R = \sum_{\alpha_{\rm r}} R^{\alpha_{\rm r}} \,. \tag{4.1}$$

A similar separation also holds for the counterterms, so that they are also labelled by an index $\alpha_{\rm r}$.

In the FKS case, for example, the $\alpha_{\rm r}$ contributions are obtained by first separating the real contribution R into the sum of all its flavour components. For each flavour component, one constructs the $\mathcal{S}$ functions, according to the procedure of section 2.4, and then multiplies it by the factors $\mathcal{S}_i$ or $\mathcal{S}_{ij}$. In the CS case, for each flavour component of the real contribution, one defines

$$\mathcal{S}_\alpha = \frac{\mathcal{D}_\alpha}{\sum_\beta \mathcal{D}_\beta} \,, \tag{4.2}$$

where α ranges in the set of dipoles $\mathcal{D}$ with the same flavour structure.

To each contribution $\alpha_{\rm r}$ we can associate an underlying n-body process, with a specific flavour structure. The association is performed as follows. If the singular region is collinear, the two collinear particles are merged into a single particle in such a way that flavour is conserved. In particular, a gg pair is merged into a g, a $q\bar{q}$ pair is merged into a g, and a qg ($\bar{q}g$) pair is merged into a q ($\bar{q}$). If the singular region is soft, the soft gluon is removed. Observe that, for non singular limits (for example, in case two quarks, or a quark and an antiquark of different flavour become collinear, or in case a quark becomes soft) the flavour structure of the underlying n-body process is undefined.

The factorization remnants also have a flavour structure. We label it with the index $\alpha_\oplus$, and also for these configurations there is an underlying n-body flavour structure. We call $\{\alpha_{\rm r}|f_b\}$ and $\{\alpha_\oplus|f_b\}$ the set of all values of the indices $\alpha_{\rm r}$ and $\alpha_\oplus$ that have the flavour structure of the underlying n-body-configuration equal to f_b.

We rewrite eq. (2.42) according to the notation of this section (making use of a straightforward extension of the context square brackets introduced in eq. (2.21))

$$\langle O\rangle = \sum_{f_b}\left[\langle O\rangle_B^{f_b} + \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}}\langle O\rangle_R^{\alpha_{\rm r}} + \sum_{\alpha_\oplus\in\{\alpha_\oplus|f_b\}}\langle O\rangle_{G_\oplus}^{\alpha_\oplus} + \sum_{\alpha_\ominus\in\{\alpha_\ominus|f_b\}}\langle O\rangle_{G_\ominus}^{\alpha_\ominus}\right], \quad (4.3)$$

$$\langle O\rangle_B^{f_b} = \int d\mathbf{\Phi}_n\, O_n(\mathbf{\Phi}_n)\,\Big[B(\mathbf{\Phi}_n) + V(\mathbf{\Phi}_n)\Big]_{f_b}, \quad (4.4)$$

$$\langle O\rangle_{G_\oplus}^{\alpha_\oplus} = \int d\mathbf{\Phi}_{n,\oplus}\, O_n(\bar{\mathbf{\Phi}}_n)\; G_\oplus^{\alpha_\oplus}(\mathbf{\Phi}_{n,\oplus}), \quad (4.5)$$

$$\langle O\rangle_R^{\alpha_{\rm r}} = \int d\mathbf{\Phi}_{n+1}\Big[O_{n+1}(\mathbf{\Phi}_{n+1})\,R(\mathbf{\Phi}_{n+1}) - O_n(\bar{\mathbf{\Phi}}_n)\,C(\mathbf{\Phi}_{n+1})\Big]_{\alpha_{\rm r}}. \quad (4.6)$$

According to ref. [4], we now perform the following manipulation

$$\langle O\rangle_R^{\alpha_{\rm r}} = \langle O\rangle_{R,n}^{\alpha_{\rm r}} + \langle O\rangle_{R,n+1}^{\alpha_{\rm r}}, \quad (4.7)$$

$$\langle O\rangle_{R,n}^{\alpha_{\rm r}} = \left[\int d\mathbf{\Phi}_{n+1}\, O_n(\bar{\mathbf{\Phi}}_n)\,\Big\{R(\mathbf{\Phi}_{n+1}) - C(\mathbf{\Phi}_{n+1})\Big\}\right]_{\alpha_{\rm r}}, \quad (4.8)$$

$$\langle O\rangle_{R,n+1}^{\alpha_{\rm r}} = \left[\int d\mathbf{\Phi}_{n+1}\, R(\mathbf{\Phi}_{n+1})\,\Big\{O_{n+1}(\mathbf{\Phi}_{n+1}) - O_n(\bar{\mathbf{\Phi}}_n)\Big\}\right]_{\alpha_{\rm r}}. \quad (4.9)$$

The term of eq. (4.9) involves real radiation. All other terms (i.e. (4.4), (4.5) and (4.8)), have n-body kinematics and, according to ref. [4], should be all lumped together into an n-body kinematics term, that was called $\bar{B}$. However, we should now carefully distinguish the contributions to $\bar{B}$ according to their flavour structure. We first rewrite eqs. (4.5), (4.8) and (4.9) in the following form

$$\langle O\rangle_{G_\oplus}^{\alpha_\oplus} = \int d\bar{\mathbf{\Phi}}_n\, O_n(\bar{\mathbf{\Phi}}_n)\,\frac{dz}{z}\,G_\oplus^{\alpha_\oplus}(\mathbf{\Phi}_{n,\oplus}), \quad (4.10)$$

$$\langle O\rangle_{R,n}^{\alpha_{\rm r}} = \left[\int d\bar{\mathbf{\Phi}}_n\, O_n(\bar{\mathbf{\Phi}}_n)\; d\Phi_{\rm rad}\,\{R(\mathbf{\Phi}_{n+1}) - C(\mathbf{\Phi}_{n+1})\}\right]_{\alpha_{\rm r}}, \quad (4.11)$$

$$\langle O\rangle_{R,n+1}^{\alpha_{\rm r}} = \left[\int d\bar{\mathbf{\Phi}}_n\, d\Phi_{\rm rad}\, R(\mathbf{\Phi}_{n+1})\,\{O_{n+1}(\mathbf{\Phi}_{n+1}) - O_n(\bar{\mathbf{\Phi}}_n)\}\right]_{\alpha_{\rm r}}. \quad (4.12)$$

According to ref. [4], we can write the $\bar{B}$ functions, one for each flavour configuration, as

$$\bar{B}^{f_b}(\mathbf{\Phi}_n) = [B(\mathbf{\Phi}_n) + V(\mathbf{\Phi}_n)]_{f_b} + \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}}\int\Big[d\Phi_{\rm rad}\,\{R(\mathbf{\Phi}_{n+1}) - C(\mathbf{\Phi}_{n+1})\}\Big]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}$$
$$+ \sum_{\alpha_\oplus\in\{\alpha_\oplus|f_b\}}\int\frac{dz}{z}\,G_\oplus^{\alpha_\oplus}(\mathbf{\Phi}_{n,\oplus}) + \sum_{\alpha_\ominus\in\{\alpha_\ominus|f_b\}}\int\frac{dz}{z}\,G_\ominus^{\alpha_\ominus}(\mathbf{\Phi}_{n,\ominus}), \quad (4.13)$$

so that

$$\int d\mathbf{\Phi}_n\, O_n(\mathbf{\Phi}_n)\;\bar{B}^{f_b}(\mathbf{\Phi}_n) = \langle O\rangle_B^{f_b} + \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}}\langle O\rangle_{R,n}^{\alpha_{\rm r}} + \sum_{\alpha_\oplus\in\{\alpha_\oplus|f_b\}}\langle O\rangle_{G_\oplus}^{\alpha_\oplus} + \sum_{\alpha_\ominus\in\{\alpha_\ominus|f_b\}}\langle O\rangle_{G_\ominus}^{\alpha_\ominus}, \quad (4.14)$$

and

$$\langle O\rangle = \sum_{f_b}\int d\mathbf{\Phi}_n\, O_n(\mathbf{\Phi}_n)\ \bar{B}^{f_b}(\mathbf{\Phi}_n)$$
$$+\sum_{\alpha_{\rm r}}\left[\int d\bar{\mathbf{\Phi}}_n\ d\Phi_{\rm rad}\, R\left(\mathbf{\Phi}_{n+1}\right)\left\{O_{n+1}\left(\mathbf{\Phi}_{n+1}\right)-O_n\left(\bar{\mathbf{\Phi}}_n\right)\right\}\right]_{\alpha_{\rm r}} . \qquad (4.15)$$

We now define the Sudakov form factors

$$\Delta^{f_b}(\mathbf{\Phi}_n,p_{\rm T}) = \exp\left\{-\sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}}\int\frac{\left[d\Phi_{\rm rad}\, R\left(\mathbf{\Phi}_{n+1}\right)\ \theta\left(k_{\rm T}(\mathbf{\Phi}_{n+1})-p_{\rm T}\right)\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}}{B^{f_b}\left(\mathbf{\Phi}_n\right)}\right\} . \qquad (4.16)$$

Notice that the identification $\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n$ is a sensible one only if the underlying n-body-process flavour structure of $\alpha_{\rm r}$ is equal to f_b. In eq. (4.16), $k_{\rm T}^{\alpha_{\rm r}}$ is a function of the kinematics variables that depends upon the particular singular region we are considering (its $\alpha_{\rm r}$ index is omitted in eq. (4.16) thanks to the context convention). For initial-state collinear singularities, we require that $k_{\rm T}$ is proportional to the transverse momentum of the emitted parton with respect to the beam axis in the collinear limit. For final-state collinear singularities, assuming that the singular region corresponds to momenta k_i and k_j becoming collinear, we take as $k_{\rm T}$ the (spatial) component of k_i (or equivalently k_j) orthogonal to the sum $\vec{k}_i+\vec{k}_j$. In the following we assume that the transverse momentum is computed in the CM frame of the colliding partons.

The factorization and renormalization scales adopted in the definition of $\bar{B}$, eq. (4.13), and in the definition of the Sudakov form factors, eq. (4.16), are different. In the definition of $\bar{B}$ one adopts a choice that is appropriate to the Born cross section. In the Sudakov exponents one must instead adopt a scale of the order of $k_{\rm T}$. In section 4.5 we show that, with the above choice of scales, the Sudakov form factor of eq. (4.16) is equal, at least to the leading logarithmic (LL) level, to the DDT [34] Sudakov form factor, and that, in some cases, with a simple prescription, one can reach NLL accuracy.

The formula for the full POWHEG cross section is

$$d\sigma = \sum_{f_b}\bar{B}^{f_b}(\mathbf{\Phi}_n)\,d\mathbf{\Phi}_n\left\{\Delta^{f_b}\left(\mathbf{\Phi}_n,p_{\rm T}^{\rm min}\right)\right.$$
$$\left.+\sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}}\frac{\left[d\Phi_{\rm rad}\ \theta\left(k_{\rm T}-p_{\rm T}^{\rm min}\right)\Delta^{f_b}(\mathbf{\Phi}_n,k_{\rm T})\ R\left(\mathbf{\Phi}_{n+1}\right)\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)}\right\}, \qquad (4.17)$$

where, for ease of notation, we have dropped the $\mathbf{\Phi}_{n+1}$ argument in $k_{\rm T}^{\alpha_{\rm r}}$. The $p_{\rm T}^{\rm min}$ value introduced here is a lower cut-off on the transverse momentum, that is needed in order to avoid to reach unphysical values of the strong coupling constant and of the parton-density functions.

As discussed at the beginning of section 2.2, in case the n-body cross section possesses singular regions, the observable O_{n+1} should vanish fast enough if $\mathbf{\Phi}_{n+1}$ approaches two singular regions at the same time. Notice that, in the POWHEG cross section given in eq. (4.17), the observable function O has disappeared, so that this restriction is no longer apparent in the formula. However, thanks to the partition of the different singular regions, it is sufficient to apply to the $\bar{B}$ function a damping factor that suppresses the regions where the n-body configuration becomes singular, in order to get a finite result. In this way, the POWHEG approach more closely resembles the standard Monte Carlo generators, where the hard leading-order matrix element for jet production is appropriately cut off in order to get a finite total cross section.

4.1 Transverse-momentum ordering

A word of caution has to be said with regard to the separation of the various singular contributions in POWHEG, in cases where the underlying n-body cross section also possesses singular regions. In standard NLO calculations, these regions are avoided by simply requiring that the physical observables one computes should be finite for the n-body term. In POWHEG, this requirement is in general not sufficient to guarantee consistency. We should also require that the $k_{\rm T}$ of the generated radiation should not be harder than all the $k_{\rm T}$'s associated with the underlying Born kinematics. More precisely, we should require that radiation with $k_{\rm T}$ larger than the smallest $k_{\rm T}$ of the underlying n-body process should be suppressed. The separation of R into contributions from the various singular regions achieves to some extent this purpose: $R^{\alpha_{\rm r}}$ is suppressed in any singular region different from $\alpha_{\rm r}$. However, consistency with the treatment of soft singularities requires that the suppression should be based upon $k_{\rm T}$, rather than, for example, virtuality. Thus, in the FKS case (for example), a most appropriate choice for the $d_i^{\oplus}$ is

$$d_i^{\oplus} = (E_i)^{2b}\, 2^b (1 \mp \cos\theta_i)^b\,, \tag{4.18}$$

$$d_{ij} = \left(\frac{E_i E_j}{E_i + E_j}\right)^{2b} 2^b\, (1 - \cos\theta_{ij})^b\ , \tag{4.19}$$

that correspond to the square of the transverse momentum to the power b, rather than the form suggested in eqs (2.78) and (2.68)

4.2 Spurious singularities in the underlying Born term

It may occour that in certain non-singular $(n + 1)$ kinematic configurations, when considering a particular singular region, the corresponding underlying Born has a singularity, not present in the R term. Consider, for example, Z+jet production in hadronic collisions. The three-body process $q\,\bar{q} \to Z + g + g$, in the kinematic configuration where the momentum of the Z is parallel to the momentum of the incoming quark, is clearly not singular, as long as there are no other collinear partons. Consider now the singular region associated with the two final-state gluons becoming collinear to each other. The corresponding underlying Born configuration is a singular one, since it corresponds to the $q\,\bar{q} \to Z + g$ process with the gluon collinear to the antiquark. In the generation of radiation, this configuration will be suppressed, due to the divergent Born denominator in the second term of eq. (4.17). If (as discussed at the end of section 4) we impose a cut on the transverse momentum of the

Z boson, also the dangerous region in the three-body configuration is cut away. Thus, in general, these configurations pose a minor problem. However, we would like to outline in the following a general way to deal with them. We call d^{α_r} a variable that measures the distance from the α_r singular region (like, for example, a power of the quantities defined in eqs. (4.18) and (4.19)). We then introduce an analogous variable $d^{\alpha_r}_{\rm ub}$ that measures the distance from the singular region of the underlying Born. We then separate

$$R^{\alpha_r} = R^{\alpha_r}_{\rm ord} + R^{\alpha_r}_{\rm inv}\,, \quad R^{\alpha_r}_{\rm ord} = R^{\alpha_r}\frac{d^{\alpha_r}_{\rm ub}}{d^{\alpha_r}+d^{\alpha_r}_{\rm ub}}\,, \quad R^{\alpha_r}_{\rm inv} = R^{\alpha_r}\frac{d^{\alpha_r}}{d^{\alpha_r}+d^{\alpha_r}_{\rm ub}}\,, \tag{4.20}$$

where "ord" and "inv" stand for "ordered" and "inverse ordered". The ordered term is dominated by $d^{\alpha_r} < d^{\alpha_r}_{\rm ub}$, and the other term has the opposite order. Only $R^{\alpha_r}_{\rm ord}$ should appear in the POWHEG formula. The $R^{\alpha_r}_{\rm inv}$ term should be generated apart, as an $(n+1)$-body process, with no Sudakov form factors. It is easy to check that, in this way, the problem is completely overcome.

4.3 NLO accuracy of the POWHEG formula

In ref. [4], it was argued that the POWHEG formula yields NLO accuracy for infrared-finite observables, and that subsequent showering by an SMC program does not spoil this conclusion. The second point is a simple consequence of the fact that no radiation with $k_{\rm T}$ larger than that generated by POWHEG is allowed in the subsequent shower, and it requires no further discussion. We wish instead to address the first point more rigorously. In other words, we want to show in detail that formula (4.17), when used to compute an infrared-safe observable, yields the correct NLO accuracy. The reader that finds this conclusion already obvious can safely skip this section.

In order to ease the notation, we will drop the $\theta\left(k_{\rm T} - p_{\rm T}^{\rm min}\right)$ factor in the POWHEG formula, always assuming that this factor is present when there is real radiation.

If we apply formula (4.17) to an infrared-safe observable O, we have

$$\begin{aligned}
\langle O\rangle &= \sum_{f_b}\int d\mathbf{\Phi}_n\,\bar{B}^{f_b}(\mathbf{\Phi}_n)\Bigg\{\Delta^{f_b}\left(\mathbf{\Phi}_n, p_{\rm T}^{\rm min}\right)O_n(\mathbf{\Phi}_n) \\
&\quad + \sum_{\alpha_r\in\{\alpha_r|f_b\}}\frac{\left[\int d\Phi_{\rm rad}\,\Delta^{f_b}(\mathbf{\Phi}_n,k_{\rm T})\;R\left(\mathbf{\Phi}_{n+1}\right)O_{n+1}(\mathbf{\Phi}_{n+1})\right]_{\alpha_r}^{\bar{\mathbf{\Phi}}_n^{\alpha_r}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)}\Bigg\} \\
&= \sum_{f_b}\int d\mathbf{\Phi}_n\bar{B}^{f_b}(\mathbf{\Phi}_n) \\
&\quad\times\Bigg\{\Bigg[\Delta^{f_b}\left(\mathbf{\Phi}_n,p_{\rm T}^{\rm min}\right) + \sum_{\alpha_r\in\{\alpha_r|f_b\}}\frac{\left[\int d\Phi_{\rm rad}\,\Delta^{f_b}(\mathbf{\Phi}_n,k_{\rm T})\;R\left(\mathbf{\Phi}_{n+1}\right)\right]_{\alpha_r}^{\bar{\mathbf{\Phi}}_n^{\alpha_r}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)}\Bigg]O_n(\mathbf{\Phi}_n) \\
&\quad + \sum_{\alpha_r\in\{\alpha_r|f_b\}}\frac{\left[\int d\Phi_{\rm rad}\,\Delta^{f_b}(\mathbf{\Phi}_n,k_{\rm T})\;R\left(\mathbf{\Phi}_{n+1}\right)\left(O_{n+1}(\mathbf{\Phi}_{n+1})-O_n(\mathbf{\Phi}_n)\right)\right]_{\alpha_r}^{\bar{\mathbf{\Phi}}_n^{\alpha_r}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)}\Bigg\},
\end{aligned} \tag{4.21}$$

where, in the second equality, we have simply added and subtracted the same term proportional to $R(\mathbf{\Phi}_{n+1})\,O_n(\mathbf{\Phi}_n)$. We now show that the term in the large squared bracket in the third member of eq. (4.21) is equal to 1. In fact

$$\begin{aligned}
\sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}} & \frac{\left[\int d\Phi_{\rm rad}\;\Delta^{f_b}(\mathbf{\Phi}_n,k_{\rm T})\;R\,(\mathbf{\Phi}_{n+1})\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)} \\
&= \int_{p_{\rm T}^{\rm min}}^{\infty} dp_{\rm T} \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}} \frac{\left[\int d\Phi_{\rm rad}\;\delta(k_{\rm T}-p_{\rm T})\,\Delta^{f_b}(\mathbf{\Phi}_n,p_{\rm T})\;R\,(\mathbf{\Phi}_{n+1})\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)} \\
&= -\int_{p_{\rm T}^{\rm min}}^{\infty} dp_{\rm T}\;\Delta^{f_b}(\mathbf{\Phi}_n,p_{\rm T})\,\frac{d}{dp_{\rm T}} \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}} \frac{\left[\int d\Phi_{\rm rad}\;\theta(k_{\rm T}-p_{\rm T})\;R\,(\mathbf{\Phi}_{n+1})\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}}{B^{f_b}(\mathbf{\Phi}_n)} \\
&= \int_{p_{\rm T}^{\rm min}}^{\infty} dp_{\rm T}\;\frac{d}{dp_{\rm T}}\Delta^{f_b}(\mathbf{\Phi}_n,p_{\rm T}) = 1-\Delta^{f_b}\!\left(\mathbf{\Phi}_n,p_{\rm T}^{\rm min}\right)\;, \qquad (4.22)
\end{aligned}$$

where we have used the fact that $\Delta^{f_b}(\mathbf{\Phi}_n,\infty)=1$. Furthermore, in the last term in the large curly bracket of eq. (4.21), small $k_{\rm T}$ values in the integral are suppressed by the $O_{n+1}(\mathbf{\Phi}_{n+1})-O_n(\mathbf{\Phi}_n)$ factor, and therefore we can replace $\Delta^{f_b}\to 1$ and $B\to\bar{B}$ up to higher orders in $\alpha_{\rm S}$. Equation (4.21) thus reduces to

$$\begin{aligned}
\langle O\rangle = \sum_{f_b}\int d\mathbf{\Phi}_n \Bigg\{ & \bar{B}^{f_b}(\mathbf{\Phi}_n)\;O_n(\mathbf{\Phi}_n) \\
&+ \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}}\left[\int d\Phi_{\rm rad}\;R\,(\mathbf{\Phi}_{n+1})\,(O_{n+1}(\mathbf{\Phi}_{n+1})-O_n(\mathbf{\Phi}_n))\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}\Bigg\}\,, \qquad (4.23)
\end{aligned}$$

up to NNLO corrections. The restriction $\theta(k_{\rm T}-p_{\rm T}^{\rm min})$ can now be dropped from the $d\Phi_{\rm rad}$ integration, its effect being suppressed by powers of $p_{\rm T}^{\rm min}$, and formula eq. (4.23) is immediately found to agree with eq. (4.15), thus concluding our proof.

4.4 Practical implementation of the POWHEG formulae

The POWHEG cross section (eq. (4.17)) looks very complex, but, in fact, from a numerical point of view, it is quite easy to implement using few well-known Monte Carlo techniques.

4.4.1 Generation of the Born variables

To begin with, we must generate a Born-like configuration (a point in the $\mathbf{\Phi}_n$ space) and a value of the index f_b, with a probability given by $\bar{B}^{f_b}(\mathbf{\Phi}_n)\,d\mathbf{\Phi}_n$. The standard Monte Carlo technique used in this cases is the hit-and-miss procedure: one finds an upper bound to the cross section, generates randomly the phase-space point, and then accepts it with a probability equal to the ratio of the value of the cross section at the given point over the upper bound value. This technique is inadequate for our case, since each evaluation of the $\bar{B}$ function requires an integration over the radiation variables. We thus proceed as follows.

For each singular region, we parametrize the radiation variables $\Phi_{\rm rad}$ in terms of a set of three variables in the unit cube, that we call $X_{\rm rad} = \left\{X^{(1)}_{\rm rad}, X^{(2)}_{\rm rad}, X^{(3)}_{\rm rad}\right\}$. Similarly, the z variable in the collinear remnants is parametrized in terms of one of these three variables, that we take to be $X^{(1)}_{\rm rad}$. We then introduce the function

$$\begin{aligned}\tilde{B}^{f_b}(\mathbf{\Phi}_n, X_{\rm rad}) = {} & \left[B\left(\mathbf{\Phi}_n\right) + V\left(\mathbf{\Phi}_n\right)\right]_{f_b} \\ & + \sum_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}} \left[\left|\frac{\partial \Phi_{\rm rad}}{\partial X_{\rm rad}}\right| \left\{R\left(\mathbf{\Phi}_{n+1}\right) - C\left(\mathbf{\Phi}_{n+1}\right)\right\}\right]^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}} = \mathbf{\Phi}_n}_{\alpha_{\rm r}} \\ & + \sum_{\alpha_\oplus\in\{\alpha_\oplus|f_b\}} \frac{1}{z}\left|\frac{\partial z}{\partial X^{(1)}_{\rm rad}}\right| G^{\alpha_\oplus}_{\oplus}\left(\mathbf{\Phi}_{n,\oplus}\right) + \sum_{\alpha_\ominus\in\{\alpha_\ominus|f_b\}} \frac{1}{z}\left|\frac{\partial z}{\partial X^{(1)}_{\rm rad}}\right| G^{\alpha_\ominus}_{\ominus}\left(\mathbf{\Phi}_{n,\ominus}\right),\end{aligned} \tag{4.24}$$

so that

$$\bar{B}^{f_b}(\mathbf{\Phi}_n) = \int_0^1 dX^{(1)}_{\rm rad} \int_0^1 dX^{(2)}_{\rm rad} \int_0^1 dX^{(3)}_{\rm rad}\, \tilde{B}^{f_b}(\mathbf{\Phi}_n, X_{\rm rad})\,, \tag{4.25}$$

and we define

$$\tilde{B}(\mathbf{\Phi}_n, X_{\rm rad}) = \sum_{f_b} \tilde{B}^{f_b}(\mathbf{\Phi}_n, X_{\rm rad})\,. \tag{4.26}$$

There are computer programs that, after performing a single integration on a given function, can efficiently generate points in the integration range, distributed according to the integrand function. One such popular program is the `BASES/SPRING` package [35]. The adaptive Monte Carlo integration routine `BASES` performs the integration of the non-negative function, and stores the necessary intermediate results. The routine `SPRING` uses these information to generate unweighted events. In our case, one integrates the $\tilde{B}(\mathbf{\Phi}_n, X_{\rm rad})$ function in the full $(\mathbf{\Phi}_n,\, X_{\rm rad})$ space using `BASES`. Then one generates $(\mathbf{\Phi}_n,\, X_{\rm rad})$ points using `SPRING`. For each generated phase-space point, one chooses an f_b value with a probability equal to $\tilde{B}^{f_b}(\mathbf{\Phi}_n, X_{\rm rad})/\tilde{B}(\mathbf{\Phi}_n, X_{\rm rad})$. At this point, the $X_{\rm rad}$ values are discarded, and one has generated the $(\mathbf{\Phi}_n,\, f_b)$ values with probability proportional to $\bar{B}^{f_b}(\mathbf{\Phi}_n)$. In essence, by doing a single $(n+1)$-body phase-space integration, one is able to generate the Born configuration with reasonable efficiency.

As already pointed out, if the Born configuration possesses singular regions, the n-body phase space should be suitably constrained, in order to avoid them. This is also the case in standard SMC programs: when one deals with cross sections with singular matrix elements, typically cross sections that include the production of a light parton, one specifies a transverse-momentum cut on its momentum, in order to get a finite total cross section. Alternatively, a weight $W(\mathbf{\Phi}_n)$ should be attached to the $\tilde{B}$ function that suppresses the n-body singular regions, so that the integral of $W \times \tilde{B}$ is finite. Events are then generated using $W \times \tilde{B}$ instead of $\tilde{B}$, and a weight $W^{-1}(\mathbf{\Phi}_n)$ should be attached to each event. For example, in order to generate a sample of $Z+$jet events, knowing that we will select jets with transverse energy greater than $E_T^{\rm cut}$, we can restrict the Born events so that $E_T > E_T^{\rm cut}/2$, E_T being the transverse energy of the radiated parton. Alternatively, we weight $\tilde{B}$ with E_T^2, attach the weight $1/E_T^2$ to the generated event, and unweight the events after the event sample (including cuts) is generated. The cut on the transverse energy of the associated jet should effectively cut off the events with low parton E_T, so that the unweighting will be really possible.

4.4.2 Generation of the hardest-radiation variables

Given the Born kinematics $(\mathbf{\Phi}_n,\, f_b)$, we must now generate the hardest-radiation configuration, characterized by $\left(\alpha_{\rm r},\, \Phi_{\rm rad}^{\alpha_{\rm r}}\right)$, with $\alpha_{\rm r} \in \{\alpha_{\rm r}|f_b\}$, with probability

$$\left[\frac{R\,(\mathbf{\Phi}_{n+1})}{B^{f_b}(\mathbf{\Phi}_n)}\,\Delta^{f_b}(\mathbf{\Phi}_n, k_{\rm T}\,(\mathbf{\Phi}_{n+1}))\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n} d\Phi_{\rm rad}^{\alpha_{\rm r}}\,. \tag{4.27}$$

The Sudakov form factor can be written as

$$\Delta^{f_b}(\mathbf{\Phi}_n, p_{\rm T}) = \prod_{\alpha_{\rm r}\in\{\alpha_{\rm r}|f_b\}} \Delta_{\alpha_{\rm r}}^{f_b}(\mathbf{\Phi}_n, p_{\rm T})\,, \tag{4.28}$$

where

$$\Delta_{\alpha_{\rm r}}^{f_b}(\mathbf{\Phi}_n, p_{\rm T}) = \exp\left\{-\left[\int d\Phi_{\rm rad}\,\frac{R\,(\mathbf{\Phi}_{n+1})}{B^{f_b}\,(\mathbf{\Phi}_n)}\,\theta\left(k_{\rm T}(\mathbf{\Phi}_{n+1}) - p_{\rm T}\right)\right]_{\alpha_{\rm r}}^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n}\right\}. \tag{4.29}$$

Under these conditions, the problem of generating the radiation variables according to eq. (4.27) can be reduced to the problem of generating them with probabilities

$$\left[\frac{R^{\alpha_{\rm r}}\,(\mathbf{\Phi}_{n+1})}{B^{f_b}(\mathbf{\Phi}_n)}\,\Delta_{\alpha_{\rm r}}^{f_b}(\mathbf{\Phi}_n, k_{\rm T}\,(\mathbf{\Phi}_{n+1}))\right]^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n} d\Phi_{\rm rad}^{\alpha_{\rm r}}\,, \tag{4.30}$$

by using the highest-bid method, illustrated in appendix B. We are thus left with the problem of generating radiation variables according to eq. (4.30) for a fixed value of $\alpha_{\rm r}$. This problem can be dealt with using the veto technique, illustrated in appendix A. In order to use this technique, we need a sufficiently simple upper bounding function

$$\left[\frac{R^{\alpha_{\rm r}}\,(\mathbf{\Phi}_{n+1})}{B^{f_b}(\mathbf{\Phi}_n)}\right]^{\bar{\mathbf{\Phi}}_n^{\alpha_{\rm r}}=\mathbf{\Phi}_n} \le F(\Phi_{\rm rad}^{\alpha_{\rm r}}, \mathbf{\Phi}_n). \tag{4.31}$$

This can be found by taking the singular limit of the left hand side of eq. (4.31), that has, in general, a form suggested by the factorization theorem, and by elementary properties of the parton densities in the case of initial-state singular regions. Once the functional form of F is guessed, its normalization is found by scanning the $\mathbf{\Phi}_{n+1}$ phase space.

Within a given subtraction method, one has typically two kinds of upper bounding functions: one for final-state radiation and one for initial-state radiation. In section 7 we illustrate explicit forms for F in the FKS and CS frameworks, for both initial- and final-state radiation.

Notice that, in general, it is not necessary to separate out all $\alpha_{\rm r}$ regions in order to apply the veto method. In many cases several regions can be group together, thus simplifying the generation algorithm (see section 7).

4.5 Sudakov form factors and NLL soft gluon resummation

The purpose of the POWHEG method is to reach NLO accuracy for inclusive quantities, and leading logarithmic (LL) accuracy for exclusive final states. In this section we address the following question: to what extent we can do better than LL accuracy in the POWHEG framework? First of all, the POWHEG method deals with the hardest emission only. Subsequent emissions are handled by the shower Monte Carlo to which POWHEG is interfaced, and will have, in general, only LL accuracy. However, exclusive observables that are especially sensitive to the hardest emission will benefit from an improved POWHEG accuracy.

In this section, we show how to improve the logarithmic accuracy of POWHEG. We show that in many cases this improvement requires very minor adjustments of the POWHEG Sudakov form factor, and that in general, NLL accuracy at least in the large N_c limit (where N_c is the number of colours) should be easy to achieve. In all this section, for ease of notation, k will denote the momentum of the soft gluon.

The results of this section can be summarized as follows:

1. The POWHEG Sudakov form factor is accurate at the LL level, provided that the strong coupling constant and the parton density functions in the Sudakov exponent are evaluated at a scale of order $k_{\rm T}^2$.

2. In case of processes involving no more than 3 coloured partons (in the initial or final state), NLL accuracy is achieved by replacing the strong coupling constant in the Sudakov exponent with
$$\alpha_{\rm S} \to A\left(\alpha_{\rm S}\left(k_{\rm T}^2\right)\right), \qquad A(\alpha_{\rm S}) = \alpha_{\rm S}\left\{1+\frac{\alpha_{\rm S}}{2\pi}\left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_{\rm A}-\frac{5}{9}n_{\rm f}\right]\right\}, \tag{4.32}$$
where the $\overline{\rm MS}$, 1-loop expression of $\alpha_{\rm S}$ should be used. Furthermore, the parton densities in the exponent must be evaluated at a scale of order $k_{\rm T}^2$. The argument of $\alpha_{\rm S}$ in eq. (4.32) can also be taken equal to a function of the radiation variable that is of order $k_{\rm T}^2$ in the soft or collinear limit, but becomes exactly equal to $k_{\rm T}^2$ in the soft *and* collinear region.

3. In case of processes involving more than 3 coloured partons, the procedure of item 2 is not sufficient to guarantee NLL accuracy. There are in fact soft (NLL) contributions that do exponentiate only in a matrix sense, so that, in order to deal with them using standard Monte Carlo techniques suited for the evaluation of ordinary exponential (like, for example, the veto method), one should diagonalize their colour structure.

4. We will show that, for processes involving more than 3 coloured partons, the correct exponentiation of the soft (NLL) contributions discussed in item 3 can be easily recovered for the dominant terms in the large-N_c limit.

In the rest of this section we will demonstrate the above points. We assume, in the following, that the reader is familiar with Sudakov resummation techniques, and in particular with ref. [36]. We also reassure the reader that, if she/he is willing to accept the above points, and is not interested in implementing the 4th point of the above list, the rest of this section can be safely skipped.

We begin by recalling the structure of Sudakov resummation of soft gluon effects in QCD, taking the notation and the results given in ref. [36]. One usually assumes that there are kinematic constraints that limit the emission of large transverse-momentum partons, called Sudakov weights u in [36]. In the POWHEG Sudakov form factor, the constraint is given by the requirement that all radiation processes have transverse momentum less than a given p_{T}, so that $u = \theta\big(p_{\mathrm{T}}^2 - k_{\mathrm{T}}^2\big)$ in our case. We now review the ingredients that build up the Sudakov resummation factors at NLL level. Radiation from a final-state massless parton i carries a factor $J_i(p_{\mathrm{T}})$, given in eq. (10) of ref. [36]

$$\log J_i(p_{\mathrm{T}}) = -4\pi \int \frac{d^4k}{(2\pi)^3}\, \delta^+\big(k^2\big)\, \theta\big(k_{\mathrm{T}}^2 - p_{\mathrm{T}}^2\big)\, \frac{\theta(z)}{k_i \cdot k}\, A\big(\alpha_{\mathrm{S}}\big(k_{\mathrm{T}}^2\big)\big)\, P_i(z), \tag{4.33}$$

where the function A is defined in eq. (4.32). Furthermore,

$$z = 1 - \frac{k \cdot n}{k_i \cdot n}, \tag{4.34}$$

where n is a timelike vector, that we take to coincide with the time direction in the partonic CM system,[15] and k_{T} is the transverse momentum of k relative to k_i, defined as

$$k_{\mathrm{T}}^2 = 2(1-z) k_i \cdot k \tag{4.35}$$

in [36]. It corresponds in the soft-collinear limit to the transverse momentum of k with respect to k_i in the CM frame of the incoming partons. P_i are appropriate combination of the Altarelli-Parisi splitting functions, defined in eq. (11) of ref. [36]. The soft (i.e. $z \to 1$) and collinear singularities in formula (4.33) give rise to logarithmic terms, with a structure (at the NLL level)

$$\log J_i(p_{\mathrm{T}}) = \sum_{k=1}^{\infty} c_k^{J,\mathrm{LL}} \alpha_{\mathrm{S}}^k\big(\mu^2\big) \log^{k+1}\frac{p_{\mathrm{T}}}{\mu} + \sum_{k=1}^{\infty} c_k^{J,\mathrm{NLL}} \alpha_{\mathrm{S}}^k\big(\mu^2\big) \log^{k}\frac{p_{\mathrm{T}}}{\mu}, \tag{4.36}$$

where μ is a reference scale of the order of the renormalization scale.

The logarithms arise in the following way. The collinear region of integration generates a $\log p_{\mathrm{T}}$. The soft region (i.e. $z \to 1$) also generates a $\log p_{\mathrm{T}}$, which arises from the $1/(1-z)$ terms in the $P_i(z)$ functions. The NLL expansion of $\alpha_{\mathrm{S}}(k_{\mathrm{T}}^2)$ in powers of α_{S} evaluated at a reference scale μ has the structure

$$\alpha_{\mathrm{S}}\big(k_{\mathrm{T}}^2\big) = \alpha_{\mathrm{S}}\big(\mu^2\big) \left[\sum_{j=0}^{\infty} c_j^{\alpha,\mathrm{LL}} \left(\alpha_{\mathrm{S}}\big(\mu^2\big) \log\frac{k_{\mathrm{T}}}{\mu}\right)^j + \sum_{j=0}^{\infty} c_j^{\alpha,\mathrm{NLL}} \alpha_{\mathrm{S}}\big(\mu^2\big) \left(\alpha_{\mathrm{S}}\big(\mu^2\big) \log\frac{k_{\mathrm{T}}}{\mu^2}\right)^j + \ldots\right], \tag{4.37}$$

and thus generates higher powers of logarithms. The coefficients $c_k^{J,\mathrm{LL}}$ in eq. (4.36) depend upon the coefficient of the $1/(1-z)$ term in P_i, and upon the $c_j^{\alpha,\mathrm{LL}}$ terms in eq. (4.37). The

[15] In ref. [36] one n is defined for each k_i, Here, for simplicity, we identify all of them.

coefficients $c_k^{J,\mathrm{NLL}}$ depend also upon the full form of P_i, since non-soft, collinear terms can generate single-log contributions, and upon the $c_j^{\alpha,\mathrm{NLL}}$ terms in eq. (4.37). Furthermore, the $\mathcal{O}(\alpha_{\mathrm{S}}^2)$ term in A also contribute to the $c^{J,\mathrm{NLL}}$ coefficients. It arises from the $z \to 1$ singular part of the Altarelli Parisi splitting function, that has the form [37]

$$P^{ff}(z) = \frac{c_f}{1-z}\alpha_{\mathrm{S}}\left\{1 + \frac{\alpha_{\mathrm{S}}}{2\pi}\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_{\mathrm{A}} - \frac{5}{9}n_{\mathrm{f}}\right]\right\}, \quad c_g = 2\,C_{\mathrm{A}}, \quad c_q = C_{\mathrm{F}}. \tag{4.38}$$

This relation holds for both the spacelike and timelike splitting function. The corresponding α_{S}^2 correction is associated with a collinear and soft singularity, and thus yields two logarithms. When combined with the $c_j^{\alpha,\mathrm{LL}}$ terms of eq. (4.37) it gives rise to NLL contributions of order $\alpha_{\mathrm{S}}^2 \log^2 p_{\mathrm{T}}/\mu$, $\alpha_{\mathrm{S}}^3 \log^3 p_{\mathrm{T}}/\mu$, and so on. We notice that the expression in eq. (4.33) can be obtained from its own $\mathcal{O}(\alpha_{\mathrm{S}}(\mu^2))$ expansion

$$\log J_i(p_{\mathrm{T}}) = -4\pi \int \frac{d^4k}{(2\pi)^3}\,\delta^+\!\left(k^2\right)\,\theta\!\left(k_{\mathrm{T}}^2 - p_{\mathrm{T}}^2\right)\,\frac{\theta(z)}{k_i \cdot k}\,\alpha_{\mathrm{S}}\!\left(\mu^2\right)\,P_i(z) + \mathcal{O}\left(\alpha_{\mathrm{S}}^2\!\left(\mu^2\right)\right) \tag{4.39}$$

provided one replaces $\alpha_{\mathrm{S}}(\mu^2) \to A(\alpha_{\mathrm{S}}(k_{\mathrm{T}}^2))$.

We now notice that the POWHEG Sudakov form factor has an exact expression of order α_{S} in the exponent. Since formula (4.39) generates the dominant $\log^2 p_{\mathrm{T}}$ and $\log p_{\mathrm{T}}$ terms, it must also be contained already in the POWHEG Sudakov exponent. It thus follows that, with the replacement of eq. (4.32) in the POWHEG Sudakov, we automatically include all LL and NLL terms of the $J(p_{\mathrm{T}})$ factors. A similar reasoning holds for the resummation of initial state emissions,[16] called Δ factors in [36], and leads to the conclusion that, besides the replacement (4.32) one also needs to evaluate the parton densities in the Sudakov exponent at a scale of order k_{T}^2, in order to achieve NLL accuracy.

The exponentiation of genuine soft, non-collinear interference contributions is more problematic. These contributions do not generally factorize in terms of the Born cross section. The real emission cross section in the soft limit has the well known structure

$$\mathcal{R} \approx 4\pi\alpha_{\mathrm{S}}\left[\sum_{i\neq j}\mathcal{B}_{ij}\frac{k_i \cdot k_j}{(k_i\cdot k)(k_j \cdot k)} + \mathcal{B}\sum_i \frac{k_i^2}{(k_i \cdot k)^2}\,C_i\right] \tag{4.40}$$

where $\mathcal{B}_{ij}$ are the colour correlated Born amplitude, defined in eq. (2.97), and C_i is the Casimir invariant of the colour representation of parton i. If parton i is massless, also collinear singularities are present in formula (4.40). They can be separated out by exploiting the techniques used to derive the results in ref. [36]. Since that technique is illustrated in unpublished notes, in the following we sketch its main points. Defining $k_i + k_j = k_{ij}$, we have

$$\frac{k_i \cdot k_j}{(k_i \cdot k)(k_j \cdot k)} = \frac{k_i \cdot k_{ij} - k_i^2}{(k_i \cdot k)(k_{ij}\cdot k)} + \frac{k_j \cdot k_{ij} - k_j^2}{(k_j \cdot k)(k_{ij} \cdot k)}. \tag{4.41}$$

In case of massless partons, for example, $k_i^2 = 0$, we can separate out the collinear component

$$\frac{k_i \cdot k_{ij}}{(k_i \cdot k)(k_{ij} \cdot k)} = \frac{1}{(k_i \cdot k)}\left[\frac{k_i \cdot k_{ij}}{(k_{ij}\cdot k)} - \frac{k_i \cdot n}{k \cdot n}\right] + \frac{1}{(k_i \cdot k)}\frac{k_i \cdot n}{k \cdot n}, \tag{4.42}$$

[16] In ref. [22] this case is discussed in details, in the framework of Z pair production in hadronic collisions.

where n is an arbitrary timelike vector. In the first term of the right hand side of eq. (4.42) there are no collinear singularities, since the factor in the square bracket vanishes when k becomes collinear to k_i. The second term in (4.42) is collinear, but being independent of j gives a contribution of the form

$$\sum_j \mathcal{B}_{ij} \frac{1}{(k_i \cdot k)} \frac{k_i \cdot n}{k \cdot n} \propto \mathcal{B} \frac{1}{(k_i \cdot k)} \frac{k_i \cdot n}{k \cdot n}, \tag{4.43}$$

factorized in terms of the Born cross section. Soft terms that factorize in terms of the Born amplitude are automatically included in the POWHEG framework, since the POWHEG exponent contains precisely the factor $\mathcal{R}/\mathcal{B}$. Not so for the interference terms. We are thus left with terms of the form

$$\sum_{\substack{i \neq j \\ k_i^2=0}} \mathcal{B}_{ij} \frac{2}{(k_i \cdot k)} \left[\frac{k_i \cdot k_{ij}}{(k_{ij} \cdot k)} - \frac{k_i \cdot n}{k \cdot n} \right] + \sum_{\substack{i \neq j \\ k_i^2 \neq 0}} \mathcal{B}_{ij} \frac{2(k_i \cdot k_{ij} - k_i^2)}{(k_i \cdot k)(k_{ij} \cdot k)}. \tag{4.44}$$

We distinguish in the sum the massless and the massive case. The $\theta(k_{\mathrm{T}} - p_{\mathrm{T}})$ function in the POWHEG Sudakov exponent, eq. (4.16), could in principle be different for the various collinear regions of the soft parton. However, since formula (4.44) does not carry collinear singularities, the dominant integration region does not require any small angles. Under this condition, k_{T} is of the order of k^0, and at the single-logarithmic level one can replace

$$\theta\left(k_{\mathrm{T}}^2 - p_{\mathrm{T}}^2\right) \Longrightarrow \theta\left(k_0^2 - p_{\mathrm{T}}^2\right). \tag{4.45}$$

In other words, the integral in the radiation variables of the generic term of formula (4.44), with a theta function constraint $\theta(k_{\mathrm{T}} - p_{\mathrm{T}})$, and k_{T} defined relative to one generic k_i yields a result of the form $a \log p_{\mathrm{T}} + b$. If we replace the $\theta(k_{\mathrm{T}} - p_{\mathrm{T}})$ with $\theta(k_0 - p_{\mathrm{T}})$ the result will become $a \log p_{\mathrm{T}} + b'$, i.e. the logarithmically enhanced term remains the same.

The angular integration of the k-dependent coefficients in eq. (4.44) yields (see appendix C)

$$I_{ij} = \int d\Omega \frac{1}{(k_i \cdot k)} \left[\frac{k_i \cdot k_{ij}}{(k_{ij} \cdot k)} - \frac{k_i \cdot n}{n \cdot k} \right] = \frac{2\pi}{k_0^2} \log \frac{(k_{ij} \cdot k_i)^2 n^2}{k_{ij}^2 (n \cdot k_i)^2} \tag{4.46}$$

if $k_i^2 = 0$, and

$$I_{ij} = \int d\Omega \frac{k_i \cdot k_{ij} - k_i^2}{(k_i \cdot k)(k_{ij} \cdot k)} = \frac{2\pi}{k_0^2} \frac{1 - \frac{k_i^2}{k_i \cdot k_{ij}}}{\beta_{ij}} \log \frac{1 + \beta_{ij}}{1 - \beta_{ij}}, \qquad \beta_{ij} = \sqrt{1 - \frac{k_i^2 k_{ij}^2}{(k_i \cdot k_{ij})^2}} \tag{4.47}$$

if $k_i^2 \neq 0$. The NLL resummed Sudakov form factor associated with these soft emissions has the form

$$\Sigma^{\mathrm{int}} = \frac{1}{|\mathcal{M}|^2} \left| \exp\left[- \int \sum_{i \neq j} \frac{dk^0}{k^0} \, \theta\left(k^0 - p_{\mathrm{T}}\right) \frac{\alpha_{\mathrm{S}}\left(k_0^2\right)}{4\pi} \sum_a T_i^a \, T_j^a \, \left(I_{ij} + I_{ji}\right) \right] \mathcal{M} \right|^2. \tag{4.48}$$

where Σ^{int} corresponds to the expression for Σ in eq. (9) of [36] (see also eqs. (14) and (15) there), excluding the J factors.[17] Here $\mathcal{M}$ is the Born matrix element, viewed as a complex

[17] In ref. [36], only the case of massless partons is considered.

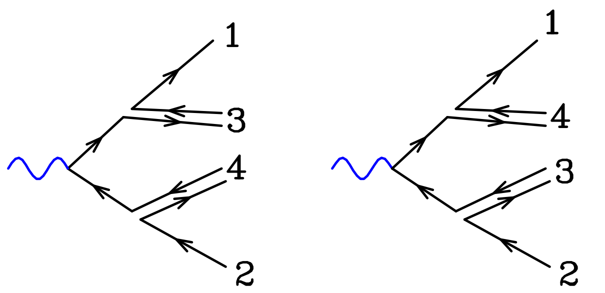

Figure 1: The two inequivalent planar colour configurations in $e^+e^- \to q\bar{q}g_3g_4$. Notice that the square of each amplitude is not symmetric in the exchange of the final state gluons.

vector in all its colour components. More precisely, $\mathcal{M}$ is a tensor in colour space, carrying a colour index for each parton entering or leaving the amplitude. It thus spans a linear space, equipped with a sesquilinear product. The matrices T_i^a can be seen as operators acting on this linear space. In facts, they act on the colour index of the ith particle in $\mathcal{M}$. Thus the exponential is to be seen as the exponential of an operator in colour space. In ref. [36] the exponential is written as an energy-ordered one, to remind that this kind of exponentiation takes place because leading-log soft emission is dominated by energy-ordered graphs, where softer emission take place later (i.e. closer to the final-state lines). In fact, at NLL accuracy the ordering has no effect.

In the POWHEG formalism, the exponentiation of the soft interference terms has the form

$$\Sigma^{\rm int} = \exp\left[-2\int\sum_{i\neq j}\frac{dk^0}{k^0}\,\theta\left(k^0-p_{\rm T}\right)\frac{\alpha_{\rm S}\left(k_0^2\right)}{4\pi}\,\frac{\mathcal{M}^\dagger\sum_a T_i^a\,T_j^a\;(I_{ij}+I_{ji})\,\mathcal{M}}{|\mathcal{M}|^2}\right], \tag{4.49}$$

since in POWHEG the ratio R/B appears in the Sudakov exponent. This does not agree in general with eq. (4.48). There is, however, one important case in which it agrees, that is to say, when $\mathcal{M}$ is an eigenstate of all the operators $\sum_a T_i^a T_j^a$. In this case we can replace $\sum_a T_i^a T_j^a$ in the exponentials in eqs. (4.48) and (4.49) with their eigenvalues, and the two expressions become identical. It is also apparent that the $\mathcal{B}_{ij}$ are in this case all proportional to $\mathcal{B}$, and again we have complete factorization of the soft contribution in terms of the Born cross section. As discussed in ref. [36], if there are no more than 3 coloured partons entering or leaving the Born amplitude, it turns out that $\mathcal{M}$ is always an eigenstate of the operator $\sum_a T_i^a T_j^a$, for any i, j. We conclude that, in this case, the prescription of eq. (4.32) is sufficient to guarantee NLL accuracy.

If there are more than 3 coloured partons in the amplitude, the standard POWHEG formula does not allow one to simply obtain full NLL accuracy. It should be very simple, however, to modify it in such a way that the interference soft terms are correctly resummed at NLL accuracy, at least as far as the dominant terms in the large-N_c limit are concerned. In the large-N_c limit, the Born amplitude can be written as the sum of independent planar colour structures, that differ only by permutations of the external lines, as illustrated in

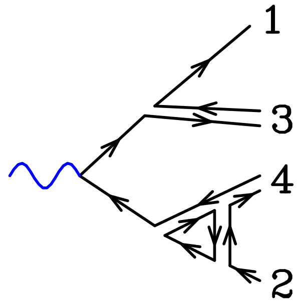

Figure 2: Soft emission colour factor $T_3^a T_4^a$ does not alter the colour structure of the Born planar amplitude, and it acts as a multiplicative constant. The close colour loop provides an extra factor of N_c.

figure 1. We write

$$\mathcal{M}_{\rm pl} = \sum_\rho \mathcal{M}_{\rm pl}^\rho, \qquad |\mathcal{M}_{\rm pl}|^2 = \sum_\rho |\mathcal{M}_{\rm pl}^\rho|^2, \tag{4.50}$$

where the subscript pl stands for planar, and ρ labels the different planar colour components. Observe that interference terms in the square of $\mathcal{M}_{\rm pl}$ do not appear, since they give rise to non-planar structures, and are suppressed in the large N_c limit. Soft emission factors give rise to leading N_c contributions only if they act upon adjacent colour lines in the planar structures. In this case their effect amounts to a factor of $N_c/2$, and they do not alter the colour connections of the Born planar amplitudes. It follows that

$$\begin{aligned}
\Sigma^{\rm int} &= \frac{1}{|\mathcal{M}_{\rm pl}|^2}\left|\exp\left[-\int \sum_{i\neq j} \frac{dk^0}{k^0}\,\theta\left(k^0 - p_{\rm T}\right)\frac{\alpha_{\rm S}\left(k_0^2\right)}{4\pi}\sum_a T_i^a T_j^a \left(I_{ij}+I_{ji}\right)\right]\mathcal{M}_{\rm pl}\right|^2 \\
&= \frac{1}{\sum_\rho |\mathcal{M}_{\rm pl}^\rho|^2}\sum_\rho |\mathcal{M}_\rho|^2 \exp\left[-2\int \sum_{i=j\pm 1}\frac{dk^0}{k^0}\,\theta\left(k^0-p_{\rm T}\right)\frac{\alpha_{\rm S}\left(k_0^2\right)}{4\pi}\frac{N_c}{2}\left(I_{ij}+I_{ji}\right)\right] \\
&= \frac{1}{\sum_\rho |\mathcal{M}_{\rm pl}^\rho|^2} \\
&\quad \times \sum_\rho |\mathcal{M}_\rho|^2 \exp\left[-2\int\sum_{i\neq j}\frac{dk^0}{k^0}\,\theta\left(k^0-p_{\rm T}\right)\frac{\alpha_{\rm S}\left(k_0^2\right)}{4\pi}\frac{\mathcal{M}_{\rm pl}^{\rho\dagger}\sum_a T_i^a T_j^a\left(I_{ij}+I_{ji}\right)\mathcal{M}_{\rm pl}^\rho}{|\mathcal{M}_{\rm pl}^\rho|^2}\right],
\end{aligned} \tag{4.51}$$

where with the notation $i = j\pm 1$ we (somewhat imprecisely) indicate that we only consider adjacent colour lines for a given planar ordered amplitude. The last line of eq. (4.51) is similar to the form that appears in eq. (4.49), except that there is one term for each planar amplitude. This suggests how to implement it in the POWHEG framework. In section 4 we have introduced a classification of the Born contributions in terms of their flavour structure f_b, and a classification of the real amplitude contributions in terms of their singularity regions and flavour structures α_r. We need to extend this classification to include also the colour structure in the large N_c limit. One simple way to do so is to compute $B_{\rm pl}$ and $R_{\rm pl}$, the Born and real terms in the planar limit, for each of their given flavour structure. Then we define

$$B^{f_b,\rho} = B^{f_b} \frac{B_{\rm pl}^{f_b,\rho}}{\sum_\rho B_{\rm pl}^{f_b,\rho}}, \qquad R^{\alpha_r,\rho_r} = R^{\alpha_r} \frac{R_{\rm pl}^{\alpha_r,\rho_r}}{\sum_{\rho_r} R_{\rm pl}^{\alpha_r,\rho_r}}, \tag{4.52}$$

so that

$$B^{f_b} = \sum_\rho B^{f_b,\rho}, \qquad R^{\alpha_r} = \sum_{\rho_r} R^{\alpha_r,\rho_r}. \tag{4.53}$$

The labels ρ and ρ_r refer to the contribution of a given planar structure in the Born and Real contributions respectively. Next we must associate with each (α_r, ρ_r) pair an underlying Born (f_b, ρ) pair. As far as f_b is concerned, the association is performed along the lines explained in section 4. For ρ and ρ_r, one proceeds as follows. If the singular region is soft, the Born planar colour structure is obtained by simply removing the soft gluon (and joining its colour lines) from the real emission colour structure. If the singular region is collinear, we need to show first that each R^{α_r,ρ_r} has collinear singularities only for collinear particles that are nearby in its planar structure. This property follows from the fact that the ratio

$$\frac{R^{\alpha_r}}{\sum_{\rho_r} R_{\rm pl}^{\alpha_r,\rho_r}} \tag{4.54}$$

has no collinear singularities, since the denominator is the large N_c limit of the numerator, and therefore it has the same singular structure. It follows then, from the second equality in (4.52), that R^{α_r,ρ_r} has the same collinear singularities of $R_{\rm pl}^{\alpha_r,\rho_r}$. Thus, since planar cross sections have collinear singularities only when nearby partons are collinear, R^{α_r,ρ_r} has the same property. Joining the planar colour of nearby partons yields a valid Born planar colour configuration. We now introduce

$$\bar{B}^{f_b,\rho} = \bar{B}^{f_b} \frac{B_{\rm pl}^{f_b,\rho}}{\sum_\rho B_{\rm pl}^{f_b,\rho}}, \tag{4.55}$$

and the corresponding POWHEG Sudakov form factor

$$\Delta^{f_b,\rho}(\mathbf{\Phi}_n, p_{\rm T}) = \exp\left\{ - \sum_{\alpha_r,\rho_r \in \{\alpha_r,\rho_r | f_b,\rho\}} \int \frac{\left[d\Phi_{\rm rad}\, R(\mathbf{\Phi}_{n+1})\, \theta(k_{\rm T} - p_{\rm T})\right]_{\alpha_r,\rho_r}^{\bar{\Phi}_n^{\alpha_r} = \Phi_n}}{B^{f_b,\rho}(\mathbf{\Phi}_n)} \right\}, \tag{4.56}$$

that should correctly generate large angle soft radiation in the large N_c limit. Equations (4.55) and (4.56) certainly satisfy the formula (4.51) for the soft interference terms in the large N_c limit. We must still show that collinear singularities are treated correctly for all N_c, i.e. that formula (4.56) is equivalent to formula (4.16) in the collinear regions. This is the case if

$$\sum_{\rho_r \in \{\alpha_r,\rho_r | f_b,\rho\}} \frac{R^{\alpha_r,\rho_r}}{B^{f_b,\rho}} = \frac{R^{\alpha_r}}{B^{f_b}} \tag{4.57}$$

in the α_r collinear region. Using eqs. (4.52) we write

$$\sum_{\rho_r \in \{\alpha_r,\rho_r | f_b,\rho\}} \frac{R^{\alpha_r,\rho_r}}{B^{f_b,\rho}} = \frac{R^{\alpha_r}}{B^{f_b}} \frac{\left[\sum_{\rho_r \in \{\alpha_r,\rho_r | f_b,\rho\}} R_{\rm pl}^{\alpha_r,\rho_r}\right] / B_{\rm pl}^{f_b,\rho}}{\left[\sum_{\rho_r} R_{\rm pl}^{\alpha_r,\rho_r}\right] / \sum_\rho B_{\rm pl}^{f_b,\rho}}. \tag{4.58}$$

The second term on the r.h.s. of eq. (4.58) yields 1 in the collinear limit. In fact, it is easy to convince oneself that, in this limit, both its numerator and denominator yield the (planar) Altarelli Parisi splitting kernel, that simplifies in the ratio. Thus $\Delta^{f_b,\rho}(\mathbf{\Phi}_n, p_{\rm T})$ becomes ρ independent in this limit, and the overall $\bar{B}^{f_b,\rho}$ factor can be summed in ρ, to yield back $\bar{B}^{f_b}$.

We conclude this section by reminding the reader that the importance of the inclusion of soft-interference terms should not be overemphasized at this stage. After all, the POWHEG formula in eq. (4.49) differs from formula (4.48) only starting at order $\alpha_{\rm S}^2$. It is thus unlikely that these terms will have important effects in a full POWHEG implementation. Nevertheless, if one wants to assess their importance, one can, to begin with, implement their large N_c resummation and study its impact.

4.6 Interfacing POWHEG to an SMC

The POWHEG algorithm generates the kinematics and flavour configuration of the hardest-emission event. The event should be fed into a SMC using the *Les Houches Interface for User Processes* [16] (LHIUP from now on). In particular, one should require that no events harder than the one generated by POWHEG should be generated by the SMC. This is achieved by setting the variable `SCALUP` of the LHIUP equal to the $k_{\rm T}$ of the POWHEG event. The LHIUP specifies how to pass the kinematics and flavour structure of the hard event to the SMC.

4.6.1 Colour assignment

The LHIUP also requires that the colour connections of the hard event (in the large N_c limit) should also be specified. POWHEG does not, in general, generate these large-N_c colour structures. They are needed (and can be generated) only if one wishes to reach large-N_c NLL accuracy of the Sudakov form factor, in events with more than 3 coloured partons at the Born level, as discussed in section 4.5. If this is not the case, the generation of the colour configuration should be performed after the POWHEG event has been generated. Here we illustrate two acceptable ways to perform colour assignment. The first (and simpler) approach is the following:

- Generate the POWHEG event in the standard way.
- Compute the different (planar) colour contributions to the Born cross section, at the kinematics of the generated underlying Born configuration.
- Pick an underlying Born colour configuration, with a probability proportional to the corresponding (planar) colour Born contribution.
- If no radiation has been generated, this is the colour structure of the event.
- If radiation has been generated, POWHEG has also generated a $\alpha_{\rm r}$ index, specifying a singular region. In this case we always assume that the emitted parton is (planar) colour-connected to the emitter. This fully specifies the planar colour structure of the event.

This method only requires the calculation of the planar colour-structures of the Born term. In the second method (which is usually implemented in MC@NLO) one computes all the planar colour contributions to R, and chooses the colour configuration with a probability proportional to the corresponding contribution.

5. POWHEG in the Frixione, Kunszt and Signer framework

In this section we construct a mapping from the $\mathbf{\Phi}_{n+1}$ phase space to the barred and radiation variables of eq. (2.28) (and its corresponding inverse mapping) for the two kinds of singular regions i (initial state) and ij (final state), that is compatible with the FKS subtraction method. These is the only missing ingredients one needs to construct a POWHEG generator in the FKS framework.

We stress that the mapping that we propose is not the only possible one. In practical examples one may find convenient to depart from this approach. It is however fully general, and as such it may be implemented once and for all in a computer code to be used for all processes.

5.1 Initial-state singularity

5.1.1 Radiation and barred variables

We first show how to construct the underlying Born (i.e. the barred variables) and the three radiation variables from the $\mathbf{\Phi}_{n+1}$ kinematics, for the case of the initial-state singularity region.

Without loss of generality, we assume that the FKS parton is the $(n+1)$th one, so that the mapping from k_i to $\bar{k}_i$ $(i = 1, \ldots, n)$ does not requires a relabeling of the momenta. In the FKS formulation, k_{n+1} is a function of ξ, $y = \cos\theta$ and ϕ (see eq. (2.43)), so that, in the centre of mass of the colliding partons, i.e. the CM of the $(x_\oplus K_\oplus + x_\ominus K_\ominus)$ system, we have

$$k^0_{n+1} = \frac{\sqrt{s}}{2}\xi\,, \qquad k_{n+1} = k^0_{n+1}\,(1, \sin\theta\,\sin\phi, \sin\theta\,\cos\phi, \cos\theta)\;. \tag{5.1}$$

We take ξ, y and ϕ as the radiation variables for the singular region, and obtain

$$\frac{d^3 k_{n+1}}{2k^0_{n+1}(2\pi)^3} = \frac{s}{(4\pi)^3}\,\xi\,d\xi\,dy\,d\phi\,. \tag{5.2}$$

We introduce the momentum

$$k_{\rm tot} = \sum_{i=1}^{n} k_i = x_\oplus K_\oplus + x_\ominus K_\ominus - k_{n+1}\,, \tag{5.3}$$

and construct a longitudinal boost $\mathbb{B}_L$ (longitudinal with respect to the incoming beams) such that $\mathbb{B}_L k_{\rm tot}$ has zero longitudinal component. Notice that $\mathbb{B}_L$ is unique, being given by a longitudinal boost with boost angle equal to minus the rapidity of $k_{\rm tot}$. Then we construct a transverse boost $\mathbb{B}_T$ such that $\mathbb{B}_T\mathbb{B}_L k_{\rm tot}$ has zero transverse momentum. We then define the barred momenta as

$$\bar{k}_i = \mathbb{B}_L^{-1}\mathbb{B}_T\mathbb{B}_L\,k_i\,, \qquad i = 1, \ldots, n\,, \tag{5.4}$$

and define

$$\bar{k}_{\rm tot} = \sum_{i=1}^{n} \bar{k}_i = \mathbb{B}_L^{-1}\mathbb{B}_T\mathbb{B}_L k_{\rm tot}\,. \tag{5.5}$$

We notice that, by construction, $k_{\rm tot}$ and $\bar{k}_{\rm tot}$ have the same invariant mass and rapidity. We now define $\bar{x}_\oplus$ and $\bar{x}_\ominus$ in such a way that

$$\bar{x}_\oplus K_\oplus + \bar{x}_\ominus K_\ominus = \bar{k}_{\rm tot}\,. \tag{5.6}$$

An explicit expression for $\bar{x}_\oplus$ and $\bar{x}_\ominus$ is easily obtained, by using the fact that $\bar{k}_{\rm tot}$ and $k_{\rm tot}$ have the same invariant mass and rapidity. We get

$$\bar{x}_\oplus = x_\oplus\sqrt{1-\xi}\,\sqrt{\frac{2-\xi(1+y)}{2-\xi(1-y)}}\,, \qquad \bar{x}_\ominus = x_\ominus\sqrt{1-\xi}\,\sqrt{\frac{2-\xi(1-y)}{2-\xi(1+y)}}\,, \tag{5.7}$$

and

$$dx_\oplus\, dx_\ominus = \frac{d\bar{x}_\oplus d\bar{x}_\ominus}{1-\xi}\,. \tag{5.8}$$

Observe that we always have $\bar{x}_\oplus \le x_\oplus$ and $\bar{x}_\ominus \le x_\ominus$. The phase space $d\mathbf{\Phi}_{n+1}$ can be rewritten as follows

$$\begin{aligned} d\mathbf{\Phi}_{n+1} &= dx_\oplus\, dx_\ominus\, (2\pi)^4\delta^4\left(x_\oplus K_\oplus + x_\ominus K_\ominus - \sum_{i=1}^{n+1} k_i\right)\prod_{i=1}^{n+1}\frac{d^3k_i}{2k_i^0(2\pi)^3} \\ &= d\bar{x}_\oplus\, d\bar{x}_\ominus\,\frac{s}{(4\pi)^3}\,\frac{\xi}{1-\xi}\,d\xi\,dy\,d\phi \\ &\quad\times(2\pi)^4\delta^4\left(x_\oplus K_\oplus + x_\ominus K_\ominus - k_{n+1} - \sum_{i=1}^{n} k_i\right)\prod_{i=1}^{n}\frac{d^3k_i}{2k_i^0(2\pi)^3} \\ &= d\bar{x}_\oplus\, d\bar{x}_\ominus\,\frac{s}{(4\pi)^3}\,\frac{\xi}{1-\xi}\,d\xi\,dy\,d\phi\,(2\pi)^4\delta^4\left(\bar{x}_\oplus K_\oplus + \bar{x}_\ominus K_\ominus - \sum_{i=1}^{n}\bar{k}_i\right)\prod_{i=1}^{n}\frac{d^3\bar{k}_i}{2\bar{k}_i^0(2\pi)^3} \\ &= \frac{s}{(4\pi)^3}\,\frac{\xi}{1-\xi}\,d\xi\,dy\,d\phi\,d\bar{\mathbf{\Phi}}_n\,, \end{aligned} \tag{5.9}$$

where we have used the boost invariance of the n-body phase space. Thus, from eq. (2.29) we obtain

$$d\Phi_{\rm rad} = \frac{s}{(4\pi)^3}\,\frac{\xi}{1-\xi}\,d\xi\,dy\,d\phi\,. \tag{5.10}$$

5.1.2 Inverse construction

We describe now the construction of the full $(n+1)$-particle phase space, given the barred variables $\bar{\mathbf{\Phi}}_n$ and the radiation variables ξ, $y = \cos\theta$ and ϕ. Using eq. (5.1) we fully construct k_{n+1}. Inverting eq. (5.7), we have

$$x_\oplus = \frac{\bar{x}_\oplus}{\sqrt{1-\xi}}\,\sqrt{\frac{2-\xi(1-y)}{2-\xi(1+y)}}, \qquad x_\ominus = \frac{\bar{x}_\ominus}{\sqrt{1-\xi}}\,\sqrt{\frac{2-\xi(1+y)}{2-\xi(1-y)}}\,. \tag{5.11}$$

The range for the variable ξ is now restricted by the requirement that both $x_\oplus$ and $x_\ominus$ be less than 1. This gives

$$0 \leq \xi \leq \xi_{\rm max} \tag{5.12}$$

with

$$\xi_{\rm max} = 1 - \max\left\{ \frac{2(1+y)\,\bar{x}_\oplus^2}{\sqrt{(1+\bar{x}_\oplus^2)^2(1-y)^2 + 16\,y\,\bar{x}_\oplus^2} + (1-y)(1-\bar{x}_\oplus^2)}, \right.$$
$$\left. \frac{2(1-y)\,\bar{x}_\ominus^2}{\sqrt{(1+\bar{x}_\ominus^2)^2(1+y)^2 - 16\,y\,\bar{x}_\ominus^2} + (1+y)(1-\bar{x}_\ominus^2)} \right\}. \tag{5.13}$$

From k_{n+1} and $x_\oplus$ we can construct $k_{\rm tot} = x_\oplus K_\oplus + x_\ominus K_\ominus - k_{n+1}$, and summing the n barred momenta, according to eq. (5.5), we can compute

$$\bar{k}_{\rm tot} = \sum_{i=1}^{n} \bar{k}_i\,. \tag{5.14}$$

The four vectors $\bar{k}_{\rm tot}$ and $k_{\rm tot}$ have the same invariant mass and rapidity by construction, since the relation between the $x_\oplus$ and the $\bar{x}_\oplus$ was obtained precisely from these conditions. We then construct the boost $\mathbb{B}_L$ such that $\mathbb{B}_L\bar{k}_{\rm tot}$ has zero rapidity. We will also have that $\mathbb{B}_L k_{\rm tot}$ has zero rapidity. Then we compute the transverse boost $\mathbb{B}_T$ such that

$$\mathbb{B}_T\,\mathbb{B}_L\,k_{\rm tot} = \mathbb{B}_L\,\bar{k}_{\rm tot}\,. \tag{5.15}$$

Finally, the momenta k_i, $i = 1,\ldots,n$ are obtained as

$$k_i = \mathbb{B}_L^{-1}\,\mathbb{B}_T^{-1}\,\mathbb{B}_L\,\bar{k}_i\,, \qquad i = 1,\ldots,n\,. \tag{5.16}$$

This completes the construction of the $(n+1)$-body phase space, starting from an underlying Born configuration and the three radiation variables.

5.2 Final-state singularity

5.2.1 Radiation and barred variables

In this section, we show how to construct the underlying Born and the three radiation variables, given the $\mathbf{\Phi}_{n+1}$ variables, in the case of final-state singularity. We assume, without loss of generality, that the singular region is associated with partons $(n+1)$ and n, that is, the $(n+1)$th parton becoming collinear to the nth parton, or becoming soft.

Our mapping is constructed in such a way that

$$\bar{x}_\oplus = x_\oplus\,, \qquad \bar{x}_\ominus = x_\ominus\,. \tag{5.17}$$

Thus, the partonic CM frame of the final-state $(n+1)$-particle system coincides with the CM frame of the n barred momenta $\bar{k}_i$, and we work in this frame from now on. In this section (and only here), we need to introduce a short-hand notation for the modulus of the space component of a four-vector in the CM frame

$$\underline{p} \equiv |\vec{p}|\,. \tag{5.18}$$

We then define

$$q = K_\oplus \, x_\oplus + K_\ominus \, x_\ominus = \sum_{i=1}^{n+1} k_i \,, \tag{5.19}$$

and, in the CM frame, we have

$$\vec{q} = 0 \,, \qquad q^2 = \left(q^0\right)^2 . \tag{5.20}$$

Introducing the four momentum

$$k = k_n + k_{n+1} \,, \tag{5.21}$$

we define the radiation variables as

$$\xi = \frac{2k^0_{n+1}}{q^0} \,, \qquad y = \frac{\vec{k}_{n+1} \cdot \vec{k}_n}{\underline{k}_{n+1}\underline{k}_n} \,, \qquad \phi = \phi\left(\vec{\eta} \times \vec{k},\ \vec{k}_{n+1} \times \vec{k}\right) , \tag{5.22}$$

where $\vec{\eta}$ is an arbitrary direction that serves as origin for the azimuthal angle of $\vec{k}_{n+1}$ around $\vec{k}$, and "$\times$" is the cross vector product. The notation $\phi(\vec{v}_1, \vec{v}_2)$ denotes the angle between $\vec{v}_1$ and $\vec{v}_2$. Thus, ϕ is the azimuth of the vector $\vec{k}_{n+1}$ around the direction $\vec{k}$. Notice that only ξ and y correspond exactly to FKS variables, since the FKS azimuth is usually defined with respect to $\vec{k}_n$ rather than $\vec{k}$. We prefer the choice (5.22) because we want to defined the mapping in such a way that the $\vec{k}$ direction (rather than the $\vec{k}_n$ one) is preserved. Our choice, however, makes only irrelevant differences in the FKS formalism, since the real-emission cross section has singular distributions only in the variables ξ and y.

We introduce the recoil four-momentum and mass

$$k_{\rm rec} = \sum_{i=1}^{n-1} k_i \,, \qquad M^2_{\rm rec} = k^2_{\rm rec} \,. \tag{5.23}$$

We have

$$k_{\rm rec} = q - k \,, \qquad \vec{k}_{\rm rec} = -\vec{k} \,. \tag{5.24}$$

We construct a boost $\mathbb{B}$ along the $\vec{k}_{\rm rec}$ direction, such that the 4-momentum $(q - \mathbb{B}\, k_{\rm rec})$ is light-like, that is to say

$$(q - \mathbb{B}\, k_{\rm rec})^2 = 0 \,. \tag{5.25}$$

The boost velocity is easily computed

$$\beta = \frac{q^2 - (k^0_{\rm rec} + \underline{k}_{\rm rec})^2}{q^2 + (k^0_{\rm rec} + \underline{k}_{\rm rec})^2} \,. \tag{5.26}$$

Since $q^0 = k^0 + k^0_{\rm rec}$, and $k^0 \geq \underline{k} = \underline{k}_{\rm rec}$, β is positive and smaller than one. Thus, the boost $\mathbb{B}$ always exists. We define the barred momenta as

$$\bar{k}_i = \mathbb{B}\, k_i, \qquad i = 1, \ldots, n-1 \,, \qquad \bar{k}_n = q - \mathbb{B}\, k_{\rm rec} \,, \tag{5.27}$$

so that they clearly satisfy momentum conservation

$$\sum_{i=1}^{n} \bar{k}_i = q \,. \tag{5.28}$$

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In order to obtain $d\Phi_{\rm rad}$, we write the $(n+1)$-body phase space as

$$\begin{aligned} d\Phi_{n+1} &= \prod_{i=1}^{n+1} \frac{d^3 k_i}{2k_i^0\,(2\pi)^3}\,(2\pi)^4\,\delta^4\left(q-\sum_{i=1}^{n+1} k_i\right) \\ &= \frac{d^3 k_{n+1}}{2k_{n+1}^0\,(2\pi)^3}\,\frac{d^3 k}{2k_n^0\,(2\pi)^3}\prod_{i=1}^{n-1}\frac{d^3 k_i}{2k_i^0\,(2\pi)^3}\,(2\pi)^4\,\delta^4\left(q-k-\sum_{i=1}^{n-1} k_i\right), \end{aligned} \tag{5.29}$$

where, in the last equality, using eq. (5.21), we have traded $\vec{k}_n$ for $\vec{k}$ as independent variable. We thus have now

$$k_n^0 = \left|\vec{k}-\vec{k}_{n+1}\right|, \qquad\qquad k^0 = k_{n+1}^0 + k_n^0\,. \tag{5.30}$$

We identify the phase space in eq. (5.29) with the phase space written in terms of the barred and radiation variables, multiplied by a Jacobian factor J

$$\begin{aligned} d\Phi_{n+1} &= \frac{d^3 k_{n+1}}{2k_{n+1}^0\,(2\pi)^3}\,\frac{d^3 k}{2k_n^0\,(2\pi)^3}\prod_{i=1}^{n-1}\frac{d^3 k_i}{2k_i^0\,(2\pi)^3}\,(2\pi)^4\,\delta^4\left(q-k-\sum_{i=1}^{n-1} k_i\right) \\ &= d\Phi_{\rm rad}\,d\bar{\Phi}_n = (J\,d\xi\,dy\,d\phi)\left[\prod_{i=1}^{n}\frac{d^3\bar{k}_i}{2\bar{k}_i^0\,(2\pi)^3}\,(2\pi)^4\,\delta^4\left(q-\sum_{i=1}^{n}\bar{k}_i\right)\right]. \end{aligned} \tag{5.31}$$

We work out this equality simplifying common factors, until we obtain an expression for J. We consider the barred and radiation variables to be functions of the unbarred variables. First of all, we remind that $\vec{k}$ and $\vec{\bar{k}}_n$ have the same direction. We thus make the replacements

$$d^3 k = d\Omega^2\,\underline{k}^2\,d\underline{k}\,, \qquad d^3\bar{k}_n = d\Omega^2\,\underline{\bar{k}}_n^2\,d\underline{\bar{k}}_n\,, \tag{5.32}$$

and cancel the common factor $d\Omega^2$ on both sides of eq. (5.31). Boost invariance of the phase-space elements and of the four-dimensional delta function also guarantees that

$$\prod_{i=1}^{n-1}\frac{d^3 k_i}{2k_i^0\,(2\pi)^3}\,(2\pi)^4\,\delta^4\left(q-k-\sum_{i=1}^{n-1} k_i\right) = \prod_{i=1}^{n-1}\frac{d^3\bar{k}_i}{2\bar{k}_i^0\,(2\pi)^3}\,(2\pi)^4\,\delta^4\left(q-\bar{k}_n-\sum_{i=1}^{n-1}\bar{k}_i\right). \tag{5.33}$$

In fact, $\vec{\bar{k}}_n$, ξ, y and ϕ are functions of $\vec{k}_{n+1}$ and $\vec{k}$ only, while $\bar{k}_1,\ldots,\bar{k}_{n-1}$ depend upon $k_1,\ldots,k_{n-1}$ via the boost $\mathbb{B}$, that depends only upon $\vec{k}_{n+1}$ and $\vec{k}$. Furthermore, according to (5.27),

$$\mathbb{B}\,(q-k) = q-\bar{k}_n\,, \tag{5.34}$$

so that eq. (5.33) is implied by boost invariance. The members of eq. (5.33) can thus be divided out on both sides of eq. (5.31). Performing also the replacement

$$\frac{d^3 k_{n+1}}{2k_{n+1}^0\,(2\pi)^3} = \frac{q^2}{(4\pi)^3}\,\xi\,d\xi\,d\cos\psi\,d\phi\,, \tag{5.35}$$

where ψ is the angle between $\vec{k}_{n+1}$ and $\vec{k}$, on the left hand side of eq. (5.31), we can also divide out $d\xi\,d\phi$. We are left with the identity

$$\frac{q^2}{(4\pi)^3}\,\xi\,d\cos\psi\,\frac{\underline{k}^2\,d\underline{k}}{k_n^0} = J dy\,\underline{\bar{k}}_n\,d\underline{\bar{k}}_n\,, \tag{5.36}$$

and, thus, we just need to express y and $\bar{\underline{k}}_n$ in terms of $\cos\psi$ and $\underline{k}$, at fixed ξ, and compute the Jacobian of this two-variable transformation. The following sequence of relations give y and $\bar{\underline{k}}_n$ as functions of $\cos\psi$ and $\underline{k}$ (at fixed ξ)

$$\begin{aligned} \underline{k}_n &= \sqrt{\underline{k}^2 + \underline{k}_{n+1}^2 - 2\,\underline{k}\,\underline{k}_{n+1}\,\cos\psi}\,, & M_{\rm rec}^2 &= \left(q^0 - \underline{k}_{n+1} - \underline{k}_n\right)^2 - \underline{k}^2\,, \\ \bar{\underline{k}}_n &= \frac{q^2 - M_{\rm rec}^2}{2\,q^0}\,, & y &= \frac{\underline{k}^2 - \underline{k}_n^2 - \underline{k}_{n+1}^2}{2\,\underline{k}_n\,\underline{k}_{n+1}}\,. \end{aligned} \tag{5.37}$$

Thus

$$\det \begin{vmatrix} \dfrac{\partial \bar{\underline{k}}_n}{\partial \underline{k}} & \dfrac{\partial y}{\partial \underline{k}} \\ \dfrac{\partial \bar{\underline{k}}_n}{\partial \cos\psi} & \dfrac{\partial y}{\partial \cos\psi} \end{vmatrix} = \frac{\underline{k}^2}{\underline{k}_n^3}\left(\underline{k}_n - \frac{k^2}{2q^0}\right)\,, \tag{5.38}$$

where

$$k^2 = 2\,\underline{k}_n\,\underline{k}_{n+1}\,(1-y)\,, \tag{5.39}$$

and we get

$$J = \frac{q^2\,\xi}{(4\pi)^3}\,\frac{\underline{k}_n^2}{\bar{\underline{k}}_n}\left(\underline{k}_n - \frac{k^2}{2q^0}\right)^{-1}\,, \tag{5.40}$$

so that

$$d\Phi_{\rm rad} = \frac{q^2\,\xi}{(4\pi)^3}\,\frac{\underline{k}_n^2}{\bar{\underline{k}}_n}\left(\underline{k}_n - \frac{k^2}{2q^0}\right)^{-1} d\xi\,dy\,d\phi\,. \tag{5.41}$$

5.2.2 Inverse construction

In this section, we describe the construction of the full $(n+1)$-particle phase space, given the barred variables $\bar{\mathbf{\Phi}}_n$ and the radiation variables ξ, y and ϕ. We immediately have

$$k_{n+1}^0 = \underline{k}_{n+1} = \xi\,\frac{q^0}{2}\,. \tag{5.42}$$

The absolute value of the three-momentum of k_n must instead be obtained by solving the equation for the energy conservation

$$\underline{k}_n + \underline{k}_{n+1} + \sqrt{\underline{k}^2 + M_{\rm rec}^2} = q^0\,, \tag{5.43}$$

where

$$\underline{k} = \sqrt{\underline{k}_n^2 + \underline{k}_{n+1}^2 + 2\,\underline{k}_n\,\underline{k}_{n+1}\,y}\,, \tag{5.44}$$

and $M_{\rm rec}^2$, according to eqs. (5.23), (5.27) and (5.28), is given by

$$M_{\rm rec}^2 = (q - \bar{k}_n)^2\,. \tag{5.45}$$

Under the obvious condition

$$k_{n+1}^0 < \frac{q^2 - M_{\rm rec}^2}{2q^0}\,, \tag{5.46}$$

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there is one and only one (positive) solution of eq. (5.43)

$$\underline{k}_n = \frac{q^2 - M_{\rm rec}^2 - 2q^0 \underline{k}_{n+1}}{2\left[q^0 - \underline{k}_{n+1}(1-y)\right]} \,. \tag{5.47}$$

Having specified the size of k_n, we construct the vectors $\vec{k}_n$ and $\vec{k}_{n+1}$ in such a way that their vector sum $\vec{k} = \vec{k}_n + \vec{k}_{n+1}$ is parallel to $\vec{\bar{k}}_n$, and that the azimuth of $\vec{k}_{n+1}$ relative to $\vec{k}$ (the given reference direction) is ϕ. Having constructed k_n and k_{n+1}, we can define $k = k_n + k_{n+1}$ and find $k_{\rm rec} = q - k$. We can now compute β according to eq. (5.26), and obtain

$$k_i = \mathbb{B}^{-1}\,\bar{k}_i, \qquad\qquad i = 1, \ldots, n-1\,. \tag{5.48}$$

Thus, the inverse mapping exists provided

$$\xi < \frac{q^2 - M_{\rm rec}^2}{q^2}\,, \tag{5.49}$$

and it is always unique.

6. POWHEG in the Catani and Seymour framework

The separation of R into singular components may require particular attention. First of all, if one uses the formula

$$R^{\alpha_{\rm r}} = \frac{\mathcal{D}_{\alpha_{\rm r}}}{\sum_{\alpha'_r} \mathcal{D}_{\alpha'_r}}\, R\,, \tag{6.1}$$

a problem may arise due to zeros in the denominator (in fact, the CS counterterms are not necessarily positive). We have not tried to prove that this can really happen. However, the problem is easily solved by using instead (for example)

$$R^{\alpha_{\rm r}} = \frac{\mathcal{D}^2_{\alpha_{\rm r}}}{\sum_{\alpha'_r} \mathcal{D}^2_{\alpha'_r}}\, R\,. \tag{6.2}$$

In case the n-body cross section possesses singular regions, as discussed at the beginning of section 2.2, a further problem may arise. Formula (6.1) may not be adequate to partition the different singular components of R. This is better seen with an example. Consider the $Z+\text{jet}$ production process. The n-body process corresponds to a $Z+l$ final state, where l is a light parton, and the $(n+1)$-body process corresponds to a $Z+l_1+l_2$ final state, where l_1 and l_2 are light partons. Consider now the counterterm corresponding to l_2 becoming collinear to an initial-state parton. It is proportional to the underlying $Z+l_1$ parton cross section. It is therefore also singular when l_1 becomes collinear to an initial-state parton, since the $Z+l_1$ cross section is singular in this limit. In standard NLO calculation, an infrared-safe observable O, that vanishes when two singular regions are approached at the same time, suppresses the singular regions of the underlying Born process in the counterterm (see eq. (2.19)). This problem is easily solved by writing

$$R^{\alpha_{\rm r}} = \frac{H\left(\bar{\mathbf{\Phi}}_n^{(\alpha_{\rm r})}\right)\mathcal{D}_{\alpha_{\rm r}}}{\sum_{\alpha'_r} H\left(\bar{\mathbf{\Phi}}_n^{(\alpha'_r)}\right)\mathcal{D}_{\alpha'_r}}\, R\,, \tag{6.3}$$

where H is a positive function that vanishes when its argument approaches an n-body singular configuration. We again have

$$\sum_{\alpha_r} R^{\alpha_r} = R\,, \tag{6.4}$$

and now R^{α_r} is singular only in the α_r region.

If one wishes to enforce transverse-momentum ordering (see section 4.1), one can first separate R as in eqs. (2.62) and (2.63), using the $\mathcal{S}$ functions defined in eqs. (2.75) and (2.80), with the d functions defined as in eqs. (4.18) and (4.19). The contributions to R corresponding to a given singular region can then be separated according to each possible spectator using a formula similar to eq. (6.3).

We now discuss the various dipole configurations of the CS approach. We use the notation introduced in ref. [28], slightly modified in order to be consistent with the notation of the rest of this paper.

6.1 Final-state singularity with final-state spectator

6.1.1 Radiation and barred variables

The labels i, j, k denote the radiator, the emitted, and the spectator partons respectively. Without loss of generality, we assume that the radiated parton j is the $(n+1)$th parton. We introduce the variable $y_{ij,k}$, that is zero in the soft and collinear limit,

$$y_{ij,k} = \frac{k_i \cdot k_j}{k_i \cdot k_j + (k_i + k_j) \cdot k_k} = \frac{2\, k_i \cdot k_j}{(k_i + k_j + k_k)^2}\,. \tag{6.5}$$

The n barred momenta, are defined as follows

$$\bar{k}_k = \frac{1}{1 - y_{ij,k}}\, k_k\,, \tag{6.6}$$

$$\bar{k}_i = \bar{k}_{ij} = k_i + k_j - \frac{y_{ij,k}}{1 - y_{ij,k}}\, k_k\,, \tag{6.7}$$

$$\bar{k}_r = k_r\,, \qquad r = 1, \ldots, n+1, \qquad r \neq i, j, k\,. \tag{6.8}$$

Notice that we have introduced the variable $\bar{k}_{ij} = \bar{k}_i$, so that $\bar{k}_k$ and $\bar{k}_{ij}$ correspond to $\tilde{p}_k$ and $\tilde{p}_{ij}$ of ref. [28]. The expression (6.5) for $y_{ij,k}$ is determined by imposing $\bar{k}_{ij}^2 = 0$, and since

$$k_i + k_j + k_k = \bar{k}_k + \bar{k}_{ij}\,, \tag{6.9}$$

momentum conservation is enforced.

Observe that, as required, the barred momenta satisfy the momentum conservation relation

$$\bar{k}_\oplus + \bar{k}_\ominus = \sum_{i=1}^{n} \bar{k}_i\,, \tag{6.10}$$

where the initial-state barred momenta are defined as

$$\bar{k}_\oplus = k_\oplus = \bar{x}_\oplus K_\oplus \tag{6.11}$$

$$\bar{k}_{\ominus} = k_{\ominus} = \bar{x}_{\ominus} K_{\ominus} \tag{6.12}$$

so that

$$\bar{x}_{\oplus} = x_{\oplus} \,. \tag{6.13}$$

In addition, we define

$$\tilde{z}_i = \frac{k_i \cdot k_k}{(k_i + k_j) \cdot k_k} = \frac{k_i \cdot \bar{k}_k}{\bar{k}_{ij} \cdot \bar{k}_k} \,, \tag{6.14}$$

that, in the collinear limit, is equal to the fraction of the momentum carried by the ith particle. We write eq. (5.20) of ref. [28] (in $d = 4$) in our notation as

$$\left[dk_i \left(\bar{k}_{ij}, \bar{k}_k\right)\right] = \frac{(2\bar{k}_{ij} \cdot \bar{k}_k)}{16\pi^2} \frac{d\phi}{(2\pi)} \, d\tilde{z}_i \, dy_{ij,k} \, \theta\big(\tilde{z}_i(1 - \tilde{z}_i)\big) \, \theta\big(y_{ij,k}(1 - y_{ij,k})\big) \, (1 - y_{ij,k}) \,, \tag{6.15}$$

where ϕ is the azimuthal angle of k_i in the centre-of-mass system of $(\bar{k}_{ij} + \bar{k}_k)$ and

$$\int d\phi = 2\pi \,. \tag{6.16}$$

The integral of a generic finite quantity F over the $(n+1)$-parton phase space can then be written as

$$\begin{aligned}
\int & dx_{\oplus} \, dx_{\ominus} \, d\Phi_{n+1} \left(x_{\oplus} K_{\oplus} + x_{\ominus} K_{\ominus}; k_1, \ldots, k_{n+1}\right) \, \mathcal{L}(x_{\oplus}, x_{\ominus}) F\left(x_{\oplus}, x_{\ominus}; k_1, \ldots, k_{n+1}\right) \\
&= \int d\bar{x}_{\oplus} \, d\bar{x}_{\ominus} \, d\Phi_n \left(\bar{x}_{\oplus} K_{\oplus} + \bar{x}_{\ominus} K_{\ominus}; \bar{k}_1, \ldots, \bar{k}_n\right) \, \left[dk_i \left(\bar{k}_{ij}, \bar{k}_k\right)\right] \, \mathcal{L}(\bar{x}_{\oplus}, \bar{x}_{\ominus}) \\
&\qquad\qquad \times F\left(x_{\oplus}, x_{\ominus}; k_1, \ldots, k_{n+1}\right) \\
&= \int d\bar{\mathbf{\Phi}}_n \, d\Phi_{\rm rad} \, \mathcal{L}(\bar{x}_{\oplus}, \bar{x}_{\ominus}) \, F\left(x_{\oplus}, x_{\ominus}; k_1, \ldots, k_{n+1}\right) \,,
\end{aligned} \tag{6.17}$$

where

$$d\Phi_{\rm rad} = \frac{(2\bar{k}_{ij} \cdot \bar{k}_k)}{16\pi^2} \frac{d\phi}{(2\pi)} \, d\tilde{z}_i \, dy_{ij,k} \, \theta\big(\tilde{z}_i(1 - \tilde{z}_i)\big) \, \theta\big(y_{ij,k}(1 - y_{ij,k})\big) \, (1 - y_{ij,k}) \,. \tag{6.18}$$

6.1.2 Inverse construction

In this section, we describe the construction of the full $(n+1)$-particle phase space, given the barred variables $\bar{\mathbf{\Phi}}_n$ and the radiation variables. The goal is to build the momenta of the emitted particle, k_j, of the emitting particle, k_i, and of the spectator particle, k_k, given $\bar{k}_{ij}$, $\bar{k}_k$ and the three radiation variable: $y_{ij,k}$, $\tilde{z}_i$ and ϕ. All the other momenta remain unchanged, according to eq. (6.8).

From eqs. (6.5), (6.14) and from momentum conservation (6.9) we have

$$\begin{aligned}
k_i \cdot k_j &= y_{ij,k} \, \bar{k}_k \cdot \bar{k}_{ij} && (6.19) \\
k_i \cdot \bar{k}_k &= \tilde{z}_i \, \bar{k}_k \cdot \bar{k}_{ij} && (6.20) \\
k_i \cdot \bar{k}_{ij} &= y_{ij,k} \, (1 - \tilde{z}_i) \, \bar{k}_k \cdot \bar{k}_{ij} \,. && (6.21)
\end{aligned}$$

We make a Lorentz boost $\mathbb{B}$ to centre-of-mass frame of $(\bar{k}_{ij} + \bar{k}_k)$, and we denote with a "prime" the momenta in this frame. We fix the z' axis parallel to $\bar{k}'_k$. In this reference frame, the boosted momenta have the following Lorentz components

$$\bar{k}'_k = E(1,0,0,1) \tag{6.22}$$
$$\bar{k}'_{ij} = E(1,0,0,-1) \tag{6.23}$$
$$k'_i = E'_i(1, \sin\theta'_i \cos\phi, \sin\theta'_i \sin\phi, \cos\theta'_i) \tag{6.24}$$

where $\bar{k}_k \cdot \bar{k}_{ij} = 2E^2$, so that

$$E = \sqrt{\frac{\bar{k}_k \cdot \bar{k}_{ij}}{2}}\,, \tag{6.25}$$

and eqs. (6.20) and (6.21), evaluated in this frame, give

$$2E^2 \tilde{z}_i = E'_i\, E \left(1 - \cos\theta'_i\right) \tag{6.26}$$
$$2E^2\, y_{ij,k}(1-\tilde{z}_i) = E'_i\, E \left(1 + \cos\theta'_i\right). \tag{6.27}$$

Solving these equations, we derive

$$E'_i = \sqrt{\frac{\bar{k}_k \cdot \bar{k}_{ij}}{2}}\left[y_{ij,k}(1-\tilde{z}_i) + \tilde{z}_i\right] \tag{6.28}$$
$$\cos\theta'_i = \frac{y_{ij,k}(1-\tilde{z}_i) - \tilde{z}_i}{y_{ij,k}(1-\tilde{z}_i) + \tilde{z}_i}\,, \tag{6.29}$$

and we have so determined all the four components of k'_i. We boost back to the original frame and we obtain

$$k_i = \mathbb{B}^{-1} k'_i\,. \tag{6.30}$$

From eq. (6.6) and from momentum conservation we can then write

$$k_k = (1 - y_{ij,k})\bar{k}_k \tag{6.31}$$
$$k_j = y_{ij,k}\bar{k}_k + \bar{k}_{ij} - k_i\,, \tag{6.32}$$

and this completes the task of building the $(n+1)$ final-state momenta.

6.2 Final-state singularity with initial-state spectator

6.2.1 Radiation and barred variables

In this case the radiator i is a final-state parton and the spectator k is an initial-state one, that we assume for definiteness to be the $\oplus$ parton. As before, without loss of generality, we assume that the radiated parton j is the $(n+1)$th parton. We introduce the variable

$$x_{ij,\oplus} = 1 - \frac{k_i \cdot k_j}{(k_i + k_j)\cdot k_\oplus}\,, \tag{6.33}$$

that approaches 1 in the soft and collinear limit, and the following n barred momenta

$$\bar{k}_i = \bar{k}_{ij} = k_i + k_j - (1 - x_{ij,\oplus})\, k_\oplus\,, \tag{6.34}$$

$$\bar{k}_r = k_r\,, \qquad r=1,\ldots,n+1, \qquad r\neq i,j\,. \tag{6.35}$$

Momentum conservation reads

$$\bar{k}_\oplus + \bar{k}_\ominus = \sum_{i=1}^{n} \bar{k}_i\,, \tag{6.36}$$

where

$$\bar{k}_\oplus = x_{ij,\oplus} k_\oplus = \bar{x}_\oplus K_\oplus\,, \tag{6.37}$$

$$\bar{k}_\ominus = k_\ominus = \bar{x}_\ominus K_\ominus\,, \tag{6.38}$$

and

$$\bar{x}_\oplus = x_{ij,\oplus}\, x_\oplus\,, \tag{6.39}$$

$$\bar{x}_\ominus = x_\ominus\,. \tag{6.40}$$

Introducing the variable

$$\tilde{z}_i = \frac{k_i\cdot k_\oplus}{(k_i+k_j)\cdot k_\oplus}\,, \tag{6.41}$$

that, in the collinear limit, is equal to the fraction of the momentum carried by the ith particle, we can write the integral of a generic finite quantity F over the $(n+1)$-parton phase space as

$$\begin{aligned}\int & dx_\oplus\, dx_\ominus\, d\Phi_{n+1}\left(x_\oplus K_\oplus + x_\ominus K_\ominus; k_1,\ldots,k_{n+1}\right)\, \mathcal{L}(x_\oplus,x_\ominus) F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\\ &= \int dx_\oplus\, dx_\ominus\, dx\, d\Phi_n\left(x\, x_\oplus K_\oplus + x_\ominus K_\ominus; \bar{k}_1\ldots\bar{k}_n\right)\left[dk_i\left(\bar{k}_{ij};k_\oplus,x\right)\right]\,\mathcal{L}(x_\oplus,x_\ominus)\\ &\qquad\times F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\,,\end{aligned} \tag{6.42}$$

where $\left[dk_i(\bar{k}_{ij};k_\oplus,x)\right]$ is given by eq. (5.48) of ref. [28] in $d=4$ dimensions

$$\left[dk_i(\bar{k}_{ij};k_\oplus,x)\right] = \frac{(2\bar{k}_{ij}\cdot k_\oplus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{z}_i\,dx_{ij,\oplus}\,\theta\big(\tilde{z}_i(1-\tilde{z}_i)\big)\,\theta\big(x(1-x)\big)\,\delta(x-x_{ij,\oplus})\,, \tag{6.43}$$

and ϕ is the azimuthal angle of k_i in the centre-of-mass system of $(\bar{k}_{ij}+k_\oplus)$.

Performing the integration in x, that fixes $x = x_{ij,\oplus}$, with the change of variable of eqs. (6.39) and (6.40), we obtain

$$\begin{aligned}\int & d\bar{x}_\oplus\, d\bar{x}_\ominus\, \frac{dx_{ij,\oplus}}{x_{ij,\oplus}}\, d\Phi_n\left(\bar{x}_\oplus K_\oplus + \bar{x}_\ominus K_\ominus; \bar{k}_1\ldots\bar{k}_n\right)\mathcal{L}\left(\frac{\bar{x}_\oplus}{x_{ij,\oplus}},\bar{x}_\ominus\right)\\ &\qquad\times\frac{(2\bar{k}_{ij}\cdot k_\oplus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{z}_i\,\theta\big(\tilde{z}_i(1-\tilde{z}_i)\big)\,\theta\big(x_{ij,\oplus}(1-x_{ij,\oplus})\big)\\ &\qquad\times\theta\left(x_{ij,\oplus}-\bar{x}_\oplus\right)F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\\ &= \int d\bar{\boldsymbol{\Phi}}_n\, d\Phi_{\rm rad}\,\mathcal{L}\left(\frac{\bar{x}_\oplus}{x_{ij,\oplus}},\bar{x}_\ominus\right)\,F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\,,\end{aligned} \tag{6.44}$$

where

$$d\Phi_{\rm rad} = \frac{(2\bar{k}_{ij}\cdot k_\oplus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{z}_i\,\frac{dx_{ij,\oplus}}{x_{ij,\oplus}}\,\theta\big(\tilde{z}_i(1-\tilde{z}_i)\big)\,\theta\big(x_{ij,\oplus}(1-x_{ij,\oplus})\big)\,\theta\left(z-\bar{x}_\oplus\right). \tag{6.45}$$

6.2.2 Inverse construction

In order to reconstruct the $(n+1)$ momenta from $\bar{\mathbf{\Phi}}_n$ and $\Phi_{\rm rad}$, we follow closely the procedure used in section 6.1.2. More precisely, we have to reconstruct k_i, k_j and $k_\oplus$ given $\bar{k}_{ij}$, $\bar{k}_\oplus$ and the radiation variables $x_{ij,\oplus}$, $\tilde{z}_i$ and ϕ.

From eq. (6.37) we immediately have

$$k_\oplus = \frac{\bar{k}_\oplus}{x_{ij,\oplus}}\,, \tag{6.46}$$

and using eqs. (6.33) and (6.41) we can write

$$\begin{aligned} k_i \cdot k_j &= (1-x_{ij,\oplus})\, k_\oplus \cdot \bar{k}_{ij} && (6.47)\\ k_i \cdot k_\oplus &= \tilde{z}_i\, k_\oplus \cdot \bar{k}_{ij} && (6.48)\\ k_i \cdot \bar{k}_{ij} &= (1-x_{ij,\oplus})\,(1-\tilde{z}_i)\; \bar{k}_k \cdot \bar{k}_{ij}\,. && (6.49) \end{aligned}$$

These equations are similar to eqs. (6.19)–(6.21) with the substitutions

$$\begin{aligned} \bar{k}_k &\leftrightarrow k_\oplus && (6.50)\\ y_{ij,k} &\leftrightarrow 1-x_{ij,\oplus} && (6.51) \end{aligned}$$

so that, in the centre-of-mass of the $(\bar{k}_{ij}+k_\oplus)$, the energy and the angle θ'_i of k_i with the $k'_\oplus$ direction are given by (see eqs. (6.28) and (6.29))

$$\begin{aligned} E'_i &= \sqrt{\frac{k_\oplus \cdot \bar{k}_{ij}}{2}}\,\left[(1-x_{ij,\oplus})(1-\tilde{z}_i)+\tilde{z}_i\right] && (6.52)\\ \cos\theta'_i &= \frac{(1-x_{ij,\oplus})(1-\tilde{z}_i)-\tilde{z}_i}{(1-x_{ij,\oplus})(1-\tilde{z}_i)+\tilde{z}_i}\,. && (6.53) \end{aligned}$$

We can boost back to the original frame and obtain k_i. The last momentum k_j is constrained by momentum conservation

$$k_j = \bar{k}_{ij} - k_i + (1-x_{ij,\oplus})\,k_\oplus\,. \tag{6.54}$$

6.3 Initial-state singularity with final-state spectator

6.3.1 Radiation and barred variables

In the case where the parton i (assumed here to be the $(n+1)$th parton) is emitted from an initial-state parton, that we take for definiteness to be the $\oplus$ one, and in the presence of a final-state spectator k, we define the following n barred final-state momenta

$$\begin{aligned} \bar{k}_k &= k_i + k_k - (1-x_{ik,\oplus})\,k_\oplus && (6.55)\\ \bar{k}_r &= k_r \qquad r=1,\ldots,n+1, \qquad r\neq i,k\,, && (6.56) \end{aligned}$$

where the variable

$$x_{ik,\oplus} = 1 - \frac{k_i\cdot k_k}{(k_i+k_k)\cdot k_\oplus} \tag{6.57}$$

approaches 1 when k_i becomes soft, and approaches z in the collinear limit, when $k_i = (1-z)\,k_\oplus$. Momentum conservation reads

$$\bar{k}_\oplus + \bar{k}_\ominus = \sum_{i=1}^{n} \bar{k}_i\,, \tag{6.58}$$

where

$$\bar{k}_\oplus = x_{ik,\oplus}\, k_\oplus = \bar{x}_\oplus K_\oplus \tag{6.59}$$

$$\bar{k}_\ominus = k_\ominus = \bar{x}_\ominus K_\ominus \tag{6.60}$$

so that

$$\bar{x}_\oplus = x_{ik,\oplus}\, x_\oplus \tag{6.61}$$

$$\bar{x}_\ominus = x_\ominus\,. \tag{6.62}$$

Introducing the variable

$$u_i = \frac{k_i \cdot k_\oplus}{(k_i + k_k)\cdot k_\oplus}\,, \tag{6.63}$$

we can write the integral of a generic finite quantity F over the $(n+1)$-parton phase space as

$$\begin{aligned}
&\int dx_\oplus\, dx_\ominus\, d\Phi_{n+1}\left(x_\oplus K_\oplus + x_\ominus K_\ominus; k_1,\ldots,k_{n+1}\right)\, \mathcal{L}(x_\oplus, x_\ominus) F\left(x_\oplus, x_\ominus; k_1,\ldots,k_{n+1}\right)\\
&\quad = \int dx_\oplus\, dx_\ominus\, dx\, d\Phi_n\left(x\, x_\oplus K_\oplus + x_\ominus K_\ominus; \bar{k}_1 \ldots \bar{k}_n\right)\left[dk_i\left(\bar{k}_k; k_\oplus, x\right)\right]\, \mathcal{L}(x_\oplus, x_\ominus)\\
&\qquad \times F\left(x_\oplus, x_\ominus; k_1,\ldots,k_{n+1}\right)\,,
\end{aligned} \tag{6.64}$$

where $\left[dk_i(\bar{k}_k; k_\oplus, x)\right]$ is given by eq. (5.72) of ref. [28] in $d=4$ dimensions

$$\left[dk_i(\bar{k}_k; k_\oplus, x)\right] = \frac{(2\bar{k}_k \cdot k_\oplus)}{16\pi^2}\,\frac{d\phi}{2\pi}\, du_i\, dx_{ik,\oplus}\, \theta\big(u_i(1-u_i)\big)\, \theta\big(x(1-x)\big)\, \delta(x - x_{ik,\oplus})\,, \tag{6.65}$$

and ϕ is the azimuthal angle of k_i in the centre-of-mass system of $(\bar{k}_k + k_\oplus)$.

Performing the integration in x, that fixes $x = x_{ik,\oplus}$, with the change of variable of eqs. (6.61) and (6.62), we obtain

$$\begin{aligned}
&\int d\bar{x}_\oplus\, d\bar{x}_\ominus\, \frac{dx_{ik,\oplus}}{x_{ik,\oplus}}\, d\Phi_n\left(\bar{x}_\oplus K_\oplus + \bar{x}_\ominus K_\ominus; \bar{k}_1 \ldots \bar{k}_n\right) \mathcal{L}\left(\frac{\bar{x}_\oplus}{x_{ik,\oplus}}, \bar{x}_\ominus\right)\\
&\qquad\qquad \times \frac{(2\bar{k}_k \cdot k_\oplus)}{16\pi^2}\,\frac{d\phi}{2\pi}\, du_i\, \theta\big(u_i(1-u_i)\big)\, \theta\big(x_{ik,\oplus}(1-x_{ik,\oplus})\big)\\
&\qquad\qquad \times \theta\left(x_{ik,\oplus} - \bar{x}_\oplus\right) F\left(x_\oplus, x_\ominus; k_1,\ldots,k_{n+1}\right)\\
&\quad = \int d\bar{\boldsymbol{\Phi}}_n\, d\Phi_{\rm rad}\, \mathcal{L}\left(\frac{\bar{x}_\oplus}{x_{ik,\oplus}}, \bar{x}_\ominus\right)\, F\left(x_\oplus, x_\ominus; k_1,\ldots,k_{n+1}\right)\,,
\end{aligned} \tag{6.66}$$

where

$$d\Phi_{\rm rad} = \frac{(2\bar{k}_k \cdot k_\oplus)}{16\pi^2}\,\frac{d\phi}{2\pi}\, du_i\, \frac{dx_{ik,\oplus}}{x_{ik,\oplus}}\, \theta\big(u_i(1-u_i)\big)\, \theta\big(x_{ik,\oplus}(1-x_{ik,\oplus})\big)\, \theta\left(x_{ik,\oplus} - \bar{x}_\oplus\right). \tag{6.67}$$

6.3.2 Inverse construction

This case is completely analogous to the one in section 6.2.2 with the substitutions

$$k_j \leftrightarrow k_k \tag{6.68}$$

$$\bar{k}_{ij} \leftrightarrow \bar{k}_k \tag{6.69}$$

$$x_{ij,\oplus} \leftrightarrow x_{ik,\oplus} \tag{6.70}$$

$$\tilde{z}_i \leftrightarrow u_i \tag{6.71}$$

so that

$$k_\oplus = \frac{\bar{k}_\oplus}{x_{ik,\oplus}} \tag{6.72}$$

$$E'_i = \sqrt{\frac{k_\oplus \cdot \bar{k}_k}{2}}\,[(1-x_{ik,\oplus})(1-u_i)+u_i] \tag{6.73}$$

$$\cos\theta'_i = = \frac{(1-x_{ik,\oplus})(1-u_i)-u_i}{(1-x_{ik,\oplus})(1-u_i)+u_i}\,. \tag{6.74}$$

where E'_i and θ'_i are the energy and the angle that the vector k'_i forms with the direction of $k'_\oplus$, in the centre-of-mass of the $(\bar{k}_k + k_\oplus)$. The four-vector k_i is obtained from k'_i with a boost back in the original frame, while k_k is constrained by momentum conservation

$$k_k = \bar{k}_k - k_i + (1-x_{ik,\oplus})\,k_\oplus\,. \tag{6.75}$$

6.4 Initial-state singularity with initial-state spectator

6.4.1 Radiation and barred variables

In the case where the parton i (that we take to be the $(n+1)$th parton) is emitted from an initial-state parton, that we take for definiteness to be the $\oplus$ one, and in the presence of an initial-state spectator, the $\ominus$ parton, we define the following n barred final-state momenta

$$\bar{k}_r^\mu = \Lambda^\mu{}_\nu(K,\bar{K})\;k_r^\nu \qquad\qquad r=1,\ldots,n+1, \qquad\qquad r\neq i\,, \tag{6.76}$$

where the boost tensor is given by

$$\Lambda^\mu{}_\nu(K,\bar{K}) = g^\mu{}_\nu - \frac{2(K+\bar{K})^\mu(K+\bar{K})_\nu}{(K+\bar{K})^2} + \frac{2\bar{K}^\mu K_\nu}{K^2}\,, \tag{6.77}$$

$$K = k_\oplus + k_\ominus - k_i = \sum_{\substack{r=1\\ r\neq i}}^{n+1} k_r\,, \tag{6.78}$$

$$\bar{K} = x_{i,\oplus\ominus}\,k_\oplus + k_\ominus = \sum_{\substack{r=1\\ r\neq i}}^{n+1} \bar{k}_r\,, \tag{6.79}$$

and

$$x_{i,\oplus\ominus} = 1 - \frac{(k_\oplus + k_\ominus)\cdot k_i}{k_\oplus \cdot k_\ominus}\,. \tag{6.80}$$

Notice that, in the soft limit, $x_{i,\oplus\ominus}$ approaches 1, while, in the collinear limit, i.e. $k_i = (1-z)\,k_\oplus$, it approaches z and that $\bar{K}^\mu = \Lambda^\mu{}_\nu(K,\bar{K})\,K^\nu$, so that $\bar{K}^2 = K^2$.

Momentum conservation reads

$$\bar{k}_\oplus + \bar{k}_\ominus = \sum_{i=1}^{n} \bar{k}_i \tag{6.81}$$

where

$$\bar{k}_\oplus = x_{i,\oplus\ominus}\,k_\oplus = \bar{x}_\oplus K_\oplus \tag{6.82}$$

$$\bar{k}_\ominus = k_\ominus = \bar{x}_\ominus K_\ominus \tag{6.83}$$

so that

$$\bar{x}_\oplus = x_{i,\oplus\ominus}\,x_\oplus \tag{6.84}$$

$$\bar{x}_\ominus = x_\ominus\,. \tag{6.85}$$

Introducing the variable

$$\tilde{v}_i = \frac{k_\oplus\cdot k_i}{k_\oplus\cdot k_\ominus} \tag{6.86}$$

we can write the integral of a generic finite quantity F over the $(n+1)$-parton phase space as

$$\begin{aligned}\int dx_\oplus\, dx_\ominus\, d\Phi_{n+1}\left(x_\oplus K_\oplus + x_\ominus K_\ominus; k_1,\ldots,k_{n+1}\right)\,\mathcal{L}(x_\oplus,x_\ominus) F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\\ = \int dx_\oplus\, dx_\ominus\, dx\, d\Phi_n\left(x\, x_\oplus K_\oplus + x_\ominus K_\ominus;\bar{k}_1\ldots\bar{k}_n\right)\left[dk_i\left(k_\oplus,k_\ominus,x\right)\right]\,\mathcal{L}(x_\oplus,x_\ominus)\\ \times F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\,,\end{aligned} \tag{6.87}$$

where $[dk_i\,(k_\oplus,k_\ominus,x)]$ is given by eq. (5.151) of ref. [28] in $d=4$ dimensions

$$[dk_i(k_\oplus,k_\ominus,x)] = \frac{(2k_\oplus\cdot k_\ominus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{v}_i\,dx_{i,\oplus\ominus}\,\theta\left(\tilde{v}_i\right)\,\theta\left(1-\frac{\tilde{v}_i}{1-x}\right)\,\theta\big(x(1-x)\big)\,\delta(x-x_{i,\oplus\ominus})\,, \tag{6.88}$$

and ϕ is the azimuthal angle of k_i in the centre-of-mass system of $(k_\oplus+k_\ominus)$ or in the laboratory frame.

Performing the integration in x, that fixes $x = x_{i,\oplus\ominus}$, with the change of variable of eqs. (6.84) and (6.85), we obtain

$$\begin{aligned}\int d\bar{x}_\oplus\, d\bar{x}_\ominus\,\frac{dx_{i,\oplus\ominus}}{x_{i,\oplus\ominus}}\,d\Phi_n\left(\bar{x}_\oplus K_\oplus + \bar{x}_\ominus K_\ominus;\bar{k}_1\ldots\bar{k}_n\right)\mathcal{L}\left(\frac{\bar{x}_\oplus}{x_{i,\oplus\ominus}},\bar{x}_\ominus\right)\\ \times\frac{(2k_\oplus\cdot k_\ominus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{v}_i\,\theta\left(\tilde{v}_i\right)\,\theta\left(1-\frac{\tilde{v}_i}{1-x_{i,\oplus\ominus}}\right)\,\theta\big(x_{i,\oplus\ominus}(1-x_{i,\oplus\ominus})\big)\\ \times\theta\left(x_{i,\oplus\ominus}-\bar{x}_\oplus\right)F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\\ = \int d\bar{\boldsymbol{\Phi}}_n\,d\Phi_{\rm rad}\,\mathcal{L}\left(\frac{\bar{x}_\oplus}{x_{i,\oplus\ominus}},\bar{x}_\ominus\right)\,F\left(x_\oplus,x_\ominus;k_1,\ldots,k_{n+1}\right)\,,\end{aligned} \tag{6.89}$$

where

$$d\Phi_{\rm rad} = \frac{(2k_\oplus\cdot k_\ominus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{v}_i\,\frac{dx_{i,\oplus\ominus}}{x_{i,\oplus\ominus}}\theta\left(\tilde{v}_i\right)\,\theta\left(1-\frac{\tilde{v}_i}{1-x_{i,\oplus\ominus}}\right)\theta\big(x_{i,\oplus\ominus}(1-x_{i,\oplus\ominus})\big)\theta\left(x_{i,\oplus\ominus}-\bar{x}_\oplus\right)\,. \tag{6.90}$$

6.4.2 Inverse construction

To reconstruct the full $(n+1)$-particle final state, given $\bar{k}_\oplus$, $\bar{k}_\ominus$ (i.e. $\bar{x}_\oplus$ and $\bar{x}_\ominus$), the n final-state momenta $\bar{k}_r$ and the three radiation variables $x_{i,\oplus\ominus}$, $\tilde{v}_i$ and ϕ, we follow the same procedure already used in section 6.1.2. From eqs. (6.82) and (6.83) we immediately have

$$k_\oplus = \frac{\bar{k}_\oplus}{x_{i,\oplus\ominus}} \tag{6.91}$$

$$k_\ominus = \bar{k}_\ominus \tag{6.92}$$

that is

$$x_\oplus = \frac{\bar{x}_\oplus}{x_{i,\oplus\ominus}}, \qquad x_\ominus = \bar{x}_\ominus\,. \tag{6.93}$$

We compute then

$$k_i \cdot k_\oplus = \tilde{v}_i\, k_\oplus \cdot k_\ominus \tag{6.94}$$

$$k_i \cdot k_\ominus = (1 - x_{i,\oplus\ominus} - \tilde{v}_i)\, k_\oplus \cdot k_\ominus \tag{6.95}$$

in the centre-of-mass system of $(k_\oplus + k_\ominus)$, and we find that the energy E_i' and angle θ_i' that the vector k_i' forms with the $k_\oplus'$ direction are given by

$$E_i' = \sqrt{\frac{k_\oplus \cdot k_\ominus}{2}}\,(1 - x_{i,\oplus\ominus}) \tag{6.96}$$

$$\cos\theta_i' = \frac{1 - 2\tilde{v}_i - x_{i,\oplus\ominus}}{1 - x_{i,\oplus\ominus}}\,. \tag{6.97}$$

We boost k_i' back in the laboratory frame and we obtain k_i. Once k_i is known, we can build the two vectors

$$K = k_\oplus + k_\ominus - k_i\,, \tag{6.98}$$

$$\bar{K} = \bar{k}_\oplus + k_\ominus\,, \tag{6.99}$$

and the inverse of the boost tensor Λ

$$\Lambda^{-1}_{\mu\nu}(K,\bar{K}) = g_{\mu\nu} - \frac{2(K+\bar{K})_\mu (K+\bar{K})_\nu}{(K+\bar{K})^2} + \frac{2K_\mu \bar{K}_\nu}{K^2}\,, \tag{6.100}$$

and compute the remaining n momenta

$$k_r = \Lambda^{-1}(K,\bar{K})\,\bar{k}_r\,, \qquad r = 1,\ldots,n\,. \tag{6.101}$$

7. Examples

In this section we show in detail how to implement the POWHEG formalism in two simple cases: $e^+e^- \to q\bar{q}$ and vector-boson production at a hadron-hadron collider, i.e. $h_\oplus h_\ominus \to V$.

7.1 $e^+e^- \to q\bar{q}$ in the Catani and Seymour formalism

Born, virtual and real corrections. We consider the scattering $e^-(p_1)\, e^+(p_2) \to \gamma^\star \to q(k_1)\, \bar{q}(k_2)\, g(k_3)$ where, in the centre-of-mass frame,

$$p_1 = \frac{q^0}{2}(1,0,0,1) \tag{7.1}$$

$$p_2 = \frac{q^0}{2}(1,0,0,-1) \tag{7.2}$$

$$k_1 = k_1^0\,(1, \sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1) \tag{7.3}$$

$$k_2 = k_2^0\,(1, \sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2)\,, \tag{7.4}$$

and define

$$q = p_1 + p_2\,, \qquad k_3 = q - k_1 - k_2\,, \qquad x_{\{1,2,3\}} = \frac{2\,q \cdot k_{\{1,2,3\}}}{q^2}\,. \tag{7.5}$$

The Born cross section B, in arbitrary units, is given by

$$B\left(\bar{k}_1, \bar{k}_2\right) \equiv B\left(\bar{\theta}_1\right) = 1 + \cos^2\bar{\theta}_1\,, \tag{7.6}$$

where $\bar{k}_1$ and $\bar{k}_2$ are the momenta of the quark and antiquark at Born level, and are given by

$$\bar{k}_1 = \frac{q^0}{2}\left(1, \sin\bar{\theta}_1 \cos\bar{\phi}_1, \sin\bar{\theta}_1 \sin\bar{\phi}_1, \cos\bar{\theta}_1\right) \tag{7.7}$$

$$\bar{k}_2 = \frac{q^0}{2}\left(1, -\sin\bar{\theta}_1 \cos\bar{\phi}_1, -\sin\bar{\theta}_1 \sin\bar{\phi}_1, -\cos\bar{\theta}_1\right). \tag{7.8}$$

In this section we drop the flavour index f_b in the Born cross section, for ease of notation. In virtue of the azimuthal symmetry of the cross section, we could fix from now on $\bar{\phi}_1 = 0$, and perform a random azimuthal rotation at the end of the generation. Here we keep it variable. The Born two-body phase-space element is given by

$$d\bar{\Phi}_2 = \frac{d^3\bar{k}_1}{2\bar{k}_1^0(2\pi)^3}\,\frac{d^3\bar{k}_2}{2\bar{k}_2^0(2\pi)^3}\,(2\pi)^4\,\delta^4(q - \bar{k}_1 - \bar{k}_2) = \frac{1}{32\pi^2}\,d\cos\bar{\theta}_1\,d\bar{\phi}_1\,, \tag{7.9}$$

so that $\bar{\boldsymbol{\Phi}}_2 = \left\{\bar{\theta}_1, \bar{\phi}_1\right\}$, while the finite soft-virtual contribution of eq. (2.107) is equal to

$$V(\bar{\theta}_1) = \frac{\alpha_{\rm S}}{\pi}\, C_{\rm F}\, B(\bar{\theta}_1)\,. \tag{7.10}$$

The real-emission term R

$$R = \frac{8\pi C_{\rm F}\alpha_{\rm S}}{q^2}\,\frac{x_1^2\,\left(1+\cos^2\theta_1\right) + x_2^2\left(1+\cos^2\,\theta_2\right)}{(1-x_1)(1-x_2)} \tag{7.11}$$

has two soft collinear regions. We call region 1 the $k_1 \cdot k_3 \to 0$ ($x_2 \to 1$) region, and region 2 the $k_2 \cdot k_3 \to 0$ ($x_1 \to 1$) region.

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Counterterms. The counterterms can be computed using eqs. (5.2) and (5.7) of ref. [28] and are given by:

1. Region 1 ($k_1 \cdot k_3 \to 0$): $\Phi_{\rm rad}^{(1)} = \{y_{13,2},\, \tilde{z}_1,\, \phi\}$

$$C^{(1)} \equiv \mathcal{D}_{13,2} = \frac{8\pi C_{\rm F}\alpha_{\rm S}}{q^2}\frac{1}{y_{13,2}}\left[\frac{2}{1-\tilde{z}_1\,(1-y_{13,2})} - (1+\tilde{z}_1)\right] B\big(\bar{k}_1,\bar{k}_2\big) \tag{7.12}$$

where

$$\bar{k}_1 = \bar{k}_{13} = q - \frac{1}{1-y_{13,2}}k_2\,, \qquad \bar{k}_2 = \frac{1}{1-y_{13,2}}k_2\,, \tag{7.13}$$

so that

$$\bar{\theta}_1 = \pi - \theta_2\,, \qquad \bar{\phi}_1 = (\pi+\phi_2) \bmod 2\pi\,, \tag{7.14}$$

since $\bar{k}_2$ is only rescaled with respect to k_2.

We can express the counterterm $C^{(1)}$ in terms of the x_1 and x_2 variables too, and we have

$$C^{(1)} = \frac{8\pi C_{\rm F}\alpha_{\rm S}}{q^2}\left[\frac{1}{1-x_2}\left(\frac{2}{2-x_1-x_2} - (1+x_1)\right) + \frac{1-x_1}{x_2}\right] B\left(\bar{k}_1,\bar{k}_2\right)\,, \tag{7.15}$$

where we have used eqs. (6.5) and (6.14)

$$y_{13,2} = 1-x_2\,, \qquad \tilde{z}_1 = \frac{x_1+x_2-1}{x_2}\,, \tag{7.16}$$

so that

$$\bar{k}_1 = q - \frac{k_2}{x_2}\,, \qquad \bar{k}_2 = \frac{k_2}{x_2}\,. \tag{7.17}$$

This way, we can use two sets of radiation variables for the first singular region: the $\{y_{13,2},\, \tilde{z}_1,\, \phi\}$ set or $\{x_1,\, x_2,\, \phi\}$ one.

2. Region 2 ($k_2 \cdot k_3 \to 0$): $\Phi_{\rm rad}^{(2)} = \{y_{23,1},\, \tilde{z}_2,\, \phi\}$

$$C^{(2)} \equiv \mathcal{D}_{23,1} = \frac{8\pi C_{\rm F}\alpha_{\rm S}}{q^2}\frac{1}{y_{23,1}}\left[\frac{2}{1-\tilde{z}_2\,(1-y_{23,1})} - (1+\tilde{z}_2)\right] B\big(\bar{k}_1,\bar{k}_2\big) \tag{7.18}$$

where

$$\bar{k}_1 = \frac{1}{1-y_{23,1}}k_1\,, \qquad \bar{k}_2 = \bar{k}_{23} = q - \frac{1}{1-y_{23,1}}k_1\,, \tag{7.19}$$

so that

$$\bar{\theta}_1 = \theta_1\,, \qquad \bar{\phi}_1 = \phi_1\,, \tag{7.20}$$

since $\bar{k}_1$ is only rescaled with respect to k_1.

In terms of the x_1 and x_2 variables, we have

$$C^{(2)} = \frac{8\pi C_{\rm F}\alpha_{\rm S}}{q^2}\left[\frac{1}{1-x_1}\left(\frac{2}{2-x_1-x_2} - (1+x_2)\right) + \frac{1-x_2}{x_1}\right] B\big(\bar{k}_1,\bar{k}_2\big) \tag{7.21}$$

where we have used eqs. (6.5) and (6.14)

$$y_{23,1} = 1 - x_1\,, \qquad \tilde{z}_2 = \frac{x_1 + x_2 - 1}{x_1}\,, \tag{7.22}$$

so that

$$\bar{k}_1 = \frac{k_1}{x_1}\,, \qquad \bar{k}_2 = \bar{k}_{23} = q - \frac{k_1}{x_1}\,. \tag{7.23}$$

This way, we can use two sets of radiation variables for the second singular region: the $\{y_{23,1},\, \tilde{z}_2,\, \phi\}$ set or $\{x_1,\, x_2,\, \phi\}$ one.

Radiation phase space and inverse construction. The radiation phase-space element of eq. (6.18) has the same functional form in both the two regions:

$$\begin{aligned} d\Phi_{\rm rad} &= \frac{q^2}{16\pi^2}\frac{d\phi}{2\pi}\, dz\, dy\, (1-y)\, \theta\big(z(1-z)\big)\, \theta\big(y(1-y)\big) && (7.24)\\ &= \frac{q^2}{16\pi^2}\frac{d\phi}{2\pi}\, dx_1\, dx_2\, \theta(1-x_1)\, \theta(1-x_2)\, \theta(x_1+x_2-1)\,, && (7.25) \end{aligned}$$

and we can use either of the two sets of radiation variables: $\{y, z, \phi\}$ or $\{x_1, x_2, \phi\}$, where we identify $y = y_{13,2}$, $z = \tilde{z}_1$ in region 1, or $y = y_{23,1}$, $z = \tilde{z}_2$ in region 2. In each of the two regions, we can express the angles θ_1 and θ_2 in the real term (7.11) in terms of the Born barred variables, $\bar{\theta}_1$ and $\bar{\phi}_1$, and in terms of the radiation variables, that here we take to be x_1, x_2 and ϕ, once all the 3 final-state momenta have been constructed, according to the procedure described in section 6.1.2. For example, considering the first singular region and the two Born variables $\bar{\theta}_1$ and $\bar{\phi}_1$ (that fix uniquely $\bar{k}_1$ and $\bar{k}_2$), we have

$$\bar{\theta}_2 = \pi - \bar{\theta}_1\,, \qquad \bar{\phi}_2 = \big(\bar{\phi}_1 + \pi\big) \bmod 2\pi\,, \tag{7.26}$$

and k_1' of eq. (6.24) is characterized by

$$\begin{aligned} E_1' &= \frac{\sqrt{q^2}}{2} x_1\,, && (7.27)\\ \cos\theta_1' &= \frac{2 - 2x_1 - 2x_2 + x_1 x_2}{x_1\, x_2}\,, && (7.28) \end{aligned}$$

and by a random angle ϕ. Observe that, in this case, no boosts are needed for the inverse construction, since the dipole rest frame coincides with the e^+e^- centre-of-mass frame. The vector k_1 is simply obtained as follows

$$k_1 = R_z(\bar{\phi}_2)\, R_y(\bar{\theta}_2)\, k_1'\,, \tag{7.29}$$

where $R_{z/y}(\alpha)$ is a rotation around the z/y direction of an angle α. The remaining two momenta are then given by

$$k_2 = x_2\, \bar{k}_2\,, \qquad k_3 = q - k_1 - k_2\,. \tag{7.30}$$

The corresponding results for the second region are obtained from the previous ones with the exchange of indexes $1 \leftrightarrow 2$.

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Generation of the Born variables. The $\bar{B}$ function of eq. (4.13) is then given by

$$\bar{B}(\bar{\theta}_1) = B(\bar{\theta}_1) + V(\bar{\theta}_1) + \int d\Phi_{\rm rad}\left[R^{(1)} - C^{(1)}\right] + \int d\Phi_{\rm rad}\left[R^{(2)} - C^{(2)}\right], \qquad (7.31)$$

where

$$R^{(1)} = R\,\frac{C^{(1)}}{C^{(1)} + C^{(2)}}\,, \qquad R^{(2)} = R\,\frac{C^{(2)}}{C^{(1)} + C^{(2)}}\,. \qquad (7.32)$$

In order to generate the two Born variables $\bar{\theta}_1$ and $\bar{\phi}_1$ distributed according to eq. (7.31), we need to construct the $\tilde{B}$ function of eq. (4.24). We then make the following change of variables

$$x_1 = X_{\rm rad}^{(1)}\,, \qquad x_2 = 1 - x_1 + x_1\,X_{\rm rad}^{(2)} \qquad \phi = 2\pi\,X_{\rm rad}^{(3)} \qquad (7.33)$$

so that

$$d\Phi_{\rm rad} = \frac{q^2}{16\pi^2} X_{\rm rad}^{(1)}\,dX_{\rm rad}^{(1)}\,dX_{\rm rad}^{(2)}\,dX_{\rm rad}^{(3)} \qquad (7.34)$$

and we obtain

$$\tilde{B}\left(\bar{\theta}_1, X_{\rm rad}\right) = B\left(\bar{\theta}_1\right) + V\left(\bar{\theta}_1\right) + \frac{q^2}{16\pi^2}\,X_{\rm rad}^{(1)}\left[\left(R^{(1)} - C^{(1)}\right) + \left(R^{(2)} - C^{(2)}\right)\right], \qquad (7.35)$$

that satisfies

$$\int_0^1 dX_{\rm rad}^{(1)} \int_0^1 dX_{\rm rad}^{(2)} \int_0^1 dX_{\rm rad}^{(3)}\,\tilde{B}(\bar{\theta}_1, X_{\rm rad}) = \bar{B}(\bar{\theta}_1)\,. \qquad (7.36)$$

The function $\tilde{B}$ can now be integrated over the full 3-body phase space, using an integrator that can generate unweighted events, like the `SPRING-BASES` package, so that one can generate the Born configurations very efficiently.

Generation of the radiation variables. Once we have generated the Born configuration distributed according to the $\bar{B}$ function, one must generate the hardest radiation. We first define the $k_{\rm T}$ of the radiation. The definition is ambiguous to some extent. The only requirements that it must satisfy is that it must be of the order of the radiation transverse momentum in the collinear limit, and that it must coincide with it in the soft-collinear limit. A suitable definition is

$$k_{\rm T}^2 = q^2 y\,x_3 = q^2 y\left[1 - z(1-y)\right]\,, \qquad (7.37)$$

where x_3 is the energy fraction of the gluon.

The Sudakov form factor of eq. (4.16) is given by

$$\Delta(\bar{\Phi}_2, p_{\rm T}) = \exp\left\{-\sum_{\alpha_r \in \{1,2\}}\left[\int d\Phi_{\rm rad}\frac{R(\bar{\theta}_1, \bar{\phi}_1, \phi, z, y)}{B(\bar{\theta}_1)}\,\theta(k_{\rm T}^2 - p_{\rm T}^2)\right]_{\alpha_r}\right\} \qquad (7.38)$$

$$= \exp\left\{-\sum_{\alpha_r \in \{1,2\}}\left[\frac{q^2}{16\pi^2}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^1 dz\int_0^1 dy\,(1-y)\,\frac{R(\bar{\theta}_1, \bar{\phi}_1, \phi, z, y)}{B(\bar{\theta}_1)}\,\theta(k_{\rm T}^2 - p_{\rm T}^2)\right]_{\alpha_r}\right\}.$$

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In order to apply the veto method of appendix A to our case, following the procedure of section 4.4.2, we need to specify the form of an upper bounding function. A suitable choice turns out to be

$$N\,\alpha_{\rm S}\big(k_{\rm T}^2\big)\,\frac{1-y}{y}\frac{1}{1-z(1-y)} \geqslant \sum_{\alpha_r\in\{1,2\}}\left[(1-y)\,\frac{R(\bar\theta_1,\bar\phi_1,\phi,z,y)}{B(\bar\theta_1)}\right]_{\alpha_r}, \tag{7.39}$$

where

$$N=\frac{32\pi C_{\rm F}}{q^2}\,. \tag{7.40}$$

We denote the integral of the upper bounding function by

$$\Delta_u(p_{\rm T})=\exp\left\{-\frac{2C_{\rm F}}{\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^1 dz\int_0^1 dy\,\alpha_{\rm S}\big(k_{\rm T}^2\big)\,\frac{1-y}{y}\frac{1}{1-z(1-y)}\theta\big(k_{\rm T}^2-p_{\rm T}^2\big)\right\}. \tag{7.41}$$

The integral in the exponent can now be easily performed

$$\Delta_u(p_{\rm T})=\exp\left\{-\frac{2C_{\rm F}}{\pi}\int_0^1 dz\int_0^1 dy\,\frac{\alpha_{\rm S}\big(k_{\rm T}^2\big)}{k_{\rm T}^2/q^2}\,(1-y)\,\theta\left(y\,(1-z(1-y))-\frac{p_{\rm T}^2}{q^2}\right)\right\}, \tag{7.42}$$

and introducing the dimensionless quantities

$$\tilde k_{\rm T}^2=\frac{k_{\rm T}^2}{q^2}=y\,[1-z(1-y)]\,,\qquad\qquad \tilde p_{\rm T}^2=\frac{p_{\rm T}^2}{q^2}\,, \tag{7.43}$$

and the integration over $d\tilde k_{\rm T}^2$, we have

$$\begin{aligned}\Delta_u(p_{\rm T}) &= \exp\left\{-\frac{2C_{\rm F}}{\pi}\int_{\tilde p_{\rm T}^2}^1 d\tilde k_{\rm T}^2\int_0^1 dz\int_0^1 dy\,(1-y)\;\delta\left(y\,(1-z(1-y))-\tilde k_{\rm T}^2\right)\frac{\alpha_{\rm S}\left(\tilde k_{\rm T}^2q^2\right)}{\tilde k_{\rm T}^2}\right\}\\ &= \exp\left\{-\frac{2C_{\rm F}}{\pi}\int_{\tilde p_{\rm T}^2}^1 d\tilde k_{\rm T}^2\frac{\alpha_{\rm S}\left(\tilde k_{\rm T}^2q^2\right)}{\tilde k_{\rm T}^2}\int_{\tilde k_{\rm T}^2}^{\tilde k_{\rm T}}\frac{dy}{y}\right\}\\ &= \exp\left\{-\frac{2C_{\rm F}}{\pi}\int_{\tilde p_{\rm T}^2}^1 d\tilde k_{\rm T}^2\,\frac{\alpha_{\rm S}\left(\tilde k_{\rm T}^2q^2\right)}{\tilde k_{\rm T}^2}\,\log\frac{1}{\tilde k_{\rm T}}\right\}.\end{aligned} \tag{7.44}$$

The integral can now be easily performed using the one-loop expression

$$\alpha_{\rm S}\big(k_{\rm T}^2\big)=\frac{1}{b_0\log\big(k_{\rm T}^2/\Lambda^2\big)}\,, \tag{7.45}$$

and we obtain

$$\Delta_u(p_{\rm T})=\exp\left\{-\frac{C_{\rm F}}{b_0\,\pi}\left[\log\frac{p_{\rm T}^2}{q^2}+\log\frac{q^2}{\Lambda^2}\,\log\frac{\log\big(q^2/\Lambda^2\big)}{\log\big(p_{\rm T}^2/\Lambda^2\big)}\right]\right\}. \tag{7.46}$$

Observe that we have a lower limit on the acceptable values of $p_{\rm T}$, since $\alpha_{\rm S}(p_{\rm T})$ must be well defined for both the two-loop and the one-loop expression of $\alpha_{\rm S}$. We thus introduce a $p_{\rm T}^{\rm min}\gtrsim\Lambda$. Summarizing:

1. Set $p_{\mathrm{T}}^{\max} = \sqrt{q^2}$, where q^2 is the maximum value of p_{T}^2, such that $\Delta_u(p_{\mathrm{T}}^{\max}) = 1$.

2. Generate r with $0 < r < 1$, and solve the equation $r = \Delta_u(p_{\mathrm{T}})/\Delta_u(p_{\mathrm{T}}^{\max})$ for p_{T}. If no solution is found, or if $p_{\mathrm{T}} < p_{\mathrm{T}}^{\min}$ no radiation is generated, and the event is returned as a $q\bar{q}$ event.

3. If a solution with $p_{\mathrm{T}} > p_{\mathrm{T}}^{\min}$ is found, the radiation variables, according to the veto technique described in appendix A, are then distributed according to

$$\frac{1-y}{y}\frac{1}{1-z(1-y)}\,\delta\big(q^2\,y\,(1-z(1-y)) - p_{\mathrm{T}}^2\big). \tag{7.47}$$

   The δ function poses no constraints on the azimuthal angle ϕ, that is then generated uniformly between 0 and 2π, but it fixes z to be

$$z = \frac{y - p_{\mathrm{T}}^2/q^2}{y(1-y)}\,, \tag{7.48}$$

   where the probability distribution of y is dy/y, so that y is uniformly distributed in $\log y$, between the range imposed by $0 \leq z \leq 1$,

$$\log\left(\frac{p_{\mathrm{T}}^2}{q^2}\right) \leq \log y \leq \frac{1}{2}\log\left(\frac{p_{\mathrm{T}}^2}{q^2}\right)\,. \tag{7.49}$$

4. Generate a uniformly-distributed random number r' in the range

$$0 < r' < N\,\alpha_{\mathrm{S}}\,\frac{1-y}{y}\frac{1}{1-z(1-y)}\,. \tag{7.50}$$

   If

$$r' < \sum_{\alpha_r \in \{1,2\}} \left[(1-y)\,\frac{R(\bar{\theta}_1, \bar{\phi}_1, \phi, z, y)}{B(\bar{\theta}_1)}\right]_{\alpha_r}\,, \tag{7.51}$$

   accept the event. Otherwise set $p_{\mathrm{T}}^{\max} = p_{\mathrm{T}}$ and go to step 2.

5. If

$$r' \geqslant \left[(1-y)\,\frac{R(\bar{\theta}_1, \bar{\phi}_1, \phi, z, y)}{B(\bar{\theta}_1)}\right]_1 \tag{7.52}$$

   then set $\alpha_r = 2$. Otherwise set $\alpha_r = 1$.

This completes the generation of the radiation variables and of the singular region α_r.

7.2 $e^+e^- \to q\bar{q}$ in the Frixione, Kunszt and Signer formalism

Virtual and real corrections. Using the notation of section 7, we write the virtual corrections, normalized as in eq. (2.92), as

$$\mathcal{V}_b = \mathcal{N}\,\frac{\alpha_{\mathrm{S}}}{2\pi}\left[-2\left(\frac{C_{\mathrm{F}}}{\epsilon^2} + \frac{3C_{\mathrm{F}}}{2\epsilon}\right) + \frac{2C_{\mathrm{F}}}{\epsilon}\log\frac{q^2}{Q^2} + \mathcal{V}_{\mathrm{fin}}\right] B\big(\bar{\theta}_1\big)\,, \tag{7.53}$$

where

$$\mathcal{V}_{\mathrm{fin}} = C_{\mathrm{F}}\left(\pi^2 - 8 + 3\log\frac{q^2}{Q^2} - \log^2\frac{q^2}{Q^2}\right) \tag{7.54}$$

and

$$B\left(\bar{\theta}_1\right) = 1 + \cos^2\bar{\theta}_1\,. \tag{7.55}$$

Setting $\xi_c = 1$ and $\delta_{\rm O} = 2$, equations (2.100) and (2.101) become

$$\mathcal{Q} = 2C_{\rm F}\left[\left(\frac{13}{2} - \frac{2\pi^2}{3}\right) - \frac{3}{2}\log\frac{q^2}{Q^2}\right], \tag{7.56}$$

$$\mathcal{I}_{12} = \frac{1}{2}\log^2\frac{q^2}{Q^2} - \frac{\pi^2}{6}\,, \tag{7.57}$$

and, using eqs. (2.97) and (2.99) we get

$$V\left(\bar{\theta}_1\right) = \frac{\alpha_{\rm S}}{2\pi}\left[\mathcal{Q} + 2\,C_{\rm F}\,\mathcal{I}_{12} + \mathcal{V}_{\rm fin}\right]B\left(\bar{\theta}_1\right), \tag{7.58}$$

where the dependence upon the scale Q^2 completely cancels. We now consider the real emission term R. Following section 2.4, we define the functions $d_{ij} = (k_i^0\,k_j^0)^a\,(1-\cos\theta_{ij})^b$ (see eq. (2.68)), so that

$$d_{31} = 2^{b-2a}\left(q^2\right)^a(x_1x_3)^{a-b}(1-x_2)^b\,, \tag{7.59}$$

$$d_{32} = 2^{b-2a}\left(q^2\right)^a(x_2x_3)^{a-b}(1-x_1)^b\,, \tag{7.60}$$

where a and b are positive real numbers. We introduce the functions $\mathcal{S}_{ij}$ (see eq. (2.75))

$$\mathcal{S}_{ij} = \frac{\frac{1}{d_{ij}}}{\frac{1}{d_{31}} + \frac{1}{d_{32}}}\,, \qquad ij \in \{31, 32\}\,. \tag{7.61}$$

Observe that there is no need to include the h factor of eq. (2.75), since only parton 3 has an associated soft singularity, and therefore we only consider the regions 31 and 32 with $h = 1$ (i.e. we do not consider the regions 13 and 23). Furthermore, the region 12 does not correspond to any singularity of R, and thus it is not included.

We define (see eq. (2.81))

$$\xi = \frac{2k_3^0}{\sqrt{q^2}} \equiv x_3\,, \qquad y_{31} = \frac{\vec{k}_3\cdot\vec{k}_1}{k_3^0\,k_1^0}\,, \qquad y_{32} = \frac{\vec{k}_3\cdot\vec{k}_2}{k_3^0\,k_2^0}\,, \tag{7.62}$$

and (see (2.85))

$$\hat{R} = \hat{R}_{31} + \hat{R}_{32}, \qquad \hat{R}_{3\{1,2\}} = \frac{1}{\xi}\left(\frac{1}{\xi}\right)_+\left(\frac{1}{1-y_{3\{1,2\}}}\right)_+\left[\left(1-y_{3\{1,2\}}\right)\xi^2\,R_{3\{1,2\}}\right], \tag{7.63}$$

where

$$R_{ij} = R\,\mathcal{S}_{ij}\,, \qquad ij \in \{31, 32\}\,. \tag{7.64}$$

The underlying Born configuration for each singular region is easily identified as the one that preserve the direction of the non-collinear parton. Thus, the underlying Born configuration for the 32 region has $\bar{k}_1 \propto k_1$, and for the 31 region has $\bar{k}_2 \propto k_2$.

Radiation phase space and inverse construction. Using eqs. (5.31) and (5.41), we can write the three-body phase space in terms of the barred and radiation variables. We work out explicitly the formulae for the 32 singular region. The 31 one is treated in the same way. We have

$$d\Phi_3 = J_{32}\, d\xi\, dy_{32}\, d\phi\, d\bar{\Phi}_2, \tag{7.65}$$

where (see eq. (7.9))

$$d\bar{\Phi}_2 = \frac{1}{32\pi^2} d\cos\bar{\theta}_1\, d\bar{\phi}_1\ . \tag{7.66}$$

In virtue of the azimuthal symmetry of the cross section, we can fix from now on $\bar{\phi}_1 = 0$, remembering that, at the end of the procedure, we should rotate the whole event by a random azimuthal angle. From eq. (5.47) we get immediately

$$x_2 = \frac{2(1-\xi)}{2-\xi(1-y_{32})} = 1 - \frac{(1+y_{32})\,\xi}{2-\xi(1-y_{32})} \tag{7.67}$$

and, using the fact that $x_1 + x_2 + \xi = 2$, we also get

$$x_1 = 1 - \frac{(1-y_{32})\,\xi\,(1-\xi)}{2-\xi(1-y_{32})}\,. \tag{7.68}$$

Given $\bar{\theta}_1$, ξ, y_{32} and ϕ we can fully reconstruct k_1, k_2 and k_3. Finally, from eq. (5.40), we get

$$J_{32} = \frac{q^2}{(4\pi)^3}\,\frac{x_2^2}{1-\xi}\,\xi\,. \tag{7.69}$$

Generation of the Born variables. The $\bar{B}$ function of eq. (4.13) is given by

$$\bar{B}(\bar{\theta}_1) = B(\bar{\theta}_1) + V(\bar{\theta}_1) + \sum_{\alpha_r \in \{32,31\}} \left[\int d\phi\, d\xi\, dy\, J\, \hat{R}(\bar{\theta}_1, \phi, \xi, y)\right]_{\alpha_r}, \tag{7.70}$$

where we are using the "context" notation introduced in eq. (2.21). We now explicitly show how to deal with the distributions, by illustrating the computation of the 32 term as an example:

$$\int d\xi\, dy_{32}\, d\phi\, J_{32}\, \hat{R}_{32} = \frac{q^2}{(4\pi)^3}\int d\xi\, dy_{32}\, d\phi \left(\frac{1}{\xi}\right)_+ \left(\frac{1}{1-y_{32}}\right)_+ (1-y_{32})\,\xi^2\, R_{32}\,\frac{x_2^2}{1-\xi}\,. \tag{7.71}$$

One can easily verify that both J_{32} and $(1-y_{32})\,\xi^2\, R_{32}$ have a finite limit for $y_{32} \to 1$ or $\xi \to 0$. We can thus expand the distributions according to their definitions (2.88), (2.89) and (2.90). Introducing, for ease of notation, the function

$$A_{32}(\bar{\theta}_1, \phi, \xi, y_{32}) = (1-y_{32})\,\xi^2\, R_{32}\,, \tag{7.72}$$

we have

$$\int_0^{2\pi} d\phi \int_0^1 d\xi \int_{-1}^1 dy_{32}\, J_{32}\, \hat{R}_{32} = \frac{q^2}{(4\pi)^3}\int_0^{2\pi} d\phi \int_0^1 d\xi \int_{-1}^1 dy_{32}\, \frac{1}{\xi}\,\frac{1}{1-y_{32}}$$

$$\times\left\{\left[\frac{x_2^2}{1-\xi}\,A_{32}\big(\bar\theta_1,\phi,\xi,y_{32}\big)-(1-\xi)\;A_{32}\big(\bar\theta_1,\phi,\xi,1\big)\right]\right.$$
$$\left.-\Big[A_{32}\big(\bar\theta_1,\phi,0,y_{32}\big)-A_{32}\big(\bar\theta_1,\phi,0,1\big)\Big]\right\}. \qquad (7.73)$$

We observe that the Jacobian of eq. (7.69) has an integrable divergence at $(\xi, y_{23}) \to (1,-1)$ region. This happens because the recoiling system is massless, so that there is no upper bound on ξ (see eq. (5.49)). In order to clarify the nature of this singularity, we introduce the parametrization

$$\xi = 1-\rho\cos\eta\,, \qquad y_{23} = -1+\rho\sin\eta\,, \qquad (7.74)$$

and from eq. (7.69) we get

$$J_{32} = \frac{q^2}{(4\pi)^3}\frac{4(1-\xi)\xi}{\left(2(1-\xi)+\xi(1+y_{32})\right)^2} = \frac{q^2}{(4\pi)^3}\,\frac{1}{\rho}\left[\frac{4\cos\eta}{(2\cos\eta+\sin\eta)^2}+\mathcal{O}(\rho)\right], \qquad (7.75)$$

diverging as $1/\rho$ in the small ρ limit. This singularity is integrable, but cannot be integrated using Monte Carlo techniques, since it yields a divergent error. On the other hand, $J_{32}\sqrt{1-\xi}\sqrt{1+y_{32}}$ has no divergence, which suggests the use of square-root importance sampling for both $1-\xi$ and $1+y_{32}$ in order to overcome the problem. We thus parametrize the integration variables as

$$\xi = 1-\left(X_{\rm rad}^{(1)}\right)^2, \qquad y_{23} = -1+2\left(X_{\rm rad}^{(2)}\right)^2, \qquad \phi = 2\pi\,X_{\rm rad}^{(3)}. \qquad (7.76)$$

The generation of the Born variables is performed along the lines of section 4.4.1. We define

$$\tilde B\big(\bar\theta_1, X_{\rm rad}\big) = \Big(B\big(\bar\theta_1\big)+V\big(\bar\theta_1\big)\Big)+16\pi\,X_{\rm rad}^{(1)}X_{\rm rad}^{(2)}\sum_{\alpha_r\in\{32,31\}}\Big[J\,\hat R\big(\bar\theta_1,\phi,\xi,y\big)\Big]_{\alpha_r}, \qquad (7.77)$$

so that

$$\int_0^1 dX_{\rm rad}^{(1)}\int_0^1 dX_{\rm rad}^{(2)}\int_0^1 dX_{\rm rad}^{(3)}\;\tilde B\big(\bar\theta_1, X_{\rm rad}\big) = \bar B\big(\bar\theta_1\big)\;. \qquad (7.78)$$

Then one performs a full integration of $\tilde B$, using an integrator that can generate unweighted events, like the `SPRING-BASES` package. After this step, one can generate Born configurations (i.e. $\bar\theta_1$) distributed according to $\bar B$.

Generation of the radiation variables. After having generated the Born configuration distributed according to the $\bar B$ function, we must generate the hardest radiation. We first define the $k_{\rm T}$ of the radiation. As we have already pointed out, the definition is ambiguous to some extent. The only requirements that it must satisfy is that it must be of the order of the radiation transverse momentum in the collinear limit, and that it must coincide with it in the soft-collinear limit. A suitable definition is

$$k_{\rm T} = \frac{q^0}{2}\sqrt{1-y^2}\,\xi\,. \qquad (7.79)$$

The no-radiation probability, according to eq. (4.16), is given by

$$\Delta\big(\bar{\theta}_1, p_{\rm T}\big) = \exp\left\{ - \sum_{\alpha_r \in \{32,31\}} \left[\int d\phi\, d\xi\, dy \frac{J\, R\big(\bar{\theta}_1, \phi, \xi, y\big)}{B\big(\bar{\theta}_1\big)} \theta(k_{\rm T} - p_{\rm T}) \right]_{\alpha_r} \right\}. \tag{7.80}$$

In order to generate the radiation variables distributed according to eq. (7.80), we make use of the veto method of appendix A. We need then to specify the form of an upper bounding function. A suitable choice turns out to be

$$U(\xi, y) = \frac{N \alpha_{\rm S}(k_{\rm T})}{(1-y^2)\,\xi} \geqslant \sum_{\alpha_r \in \{32,31\}} \left[\frac{J\, R\big(\bar{\theta}_1, \phi, \xi, y\big)}{B\big(\bar{\theta}_1\big)} \right]_{\alpha_r} , \tag{7.81}$$

where y stands for the common value of y_{31} and y_{32} on the right hand side. Notice that $U(\xi, y)$ is singular also for $y \to -1$. In this the upper bound properly covers also the singularity in J for $y \to -1$, (see eq. (7.75)). The normalization N is evaluated by probing the $\bar{\theta}_1, \phi, \xi, y$ phase space randomly with an adequate number of points. The $\alpha_{\rm S}(k_{\rm T})$ factor appearing in eq. (7.81) corresponds to the one-loop expression.[18]

We now want to generate ϕ, ξ, y according to the distribution

$$\exp\left[- \int d\phi'\, d\xi'\, dy'\, U(\xi', y')\, \theta\big(k'_{\rm T} - k_{\rm T}\big) \right] U(\xi, y)\, d\phi\, d\xi\, dy\,, \tag{7.82}$$

where $k'_{\rm T} = k_{\rm T}(\xi', y')$ and $k_{\rm T} = k_{\rm T}(\xi, y)$, and then use the veto technique of appendix A to correct to the exact distribution. One generates first the $k_{\rm T}$ value. Its probability distribution is given by

$$\int \exp\left[- \int d\phi'\, d\xi'\, dy'\, U(\xi', y')\, \theta\big(k'_{\rm T} - k_{\rm T}\big) \right] U(\xi, y)\, d\phi\, d\xi\, dy\, \delta(k_{\rm T} - p_{\rm T}) = -\frac{d\Delta_u(p_{\rm T})}{dp_{\rm T}}\,, \tag{7.83}$$

where

$$\Delta_u(p_{\rm T}) = \exp\left[- \int d\phi\, d\xi\, dy\, U(\xi, y)\, \theta(k_{\rm T} - p_{\rm T}) \right]. \tag{7.84}$$

The integral in the exponent of eq. (7.84) is easily manipulated to yield

$$\int d\phi\, d\xi\, dy\, \frac{N \alpha_{\rm S}(k_{\rm T})}{(1-y^2)\ \xi}\, \theta(k_{\rm T} - p_{\rm T}) = 2\pi N \int_{p_{\rm T}}^{k_{\rm max}} \frac{dk_{\rm T}}{k_{\rm T}}\, \alpha_{\rm S}(k_{\rm T})\, \log \frac{1 + \sqrt{1 - (k_{\rm T}/k_{\rm max})^2}}{1 - \sqrt{1 - (k_{\rm T}/k_{\rm max})^2}}\,, \tag{7.85}$$

with

$$k_{\rm max} = q^0/2\,. \tag{7.86}$$

So, we want to generate $p_{\rm T}$ according to the Sudakov form factor

$$\Delta_u(p_{\rm T}) = \exp\left[-2\pi N \int_0^{k_{\rm max}} \frac{dk_{\rm T}}{k_{\rm T}}\, \alpha_{\rm S}(k_{\rm T})\, \log \frac{1 + \sqrt{1 - (k_{\rm T}/k_{\rm max})^2}}{1 - \sqrt{1 - (k_{\rm T}/k_{\rm max})^2}}\, \theta(k_{\rm T} - p_{\rm T}) \right], \tag{7.87}$$

[18] Since the $e^+e^- \to q\bar{q}$ cross section is of zeroth order in the strong coupling constant, the use of one-loop $\alpha_{\rm S}$ in the radiative corrections is adequate in this case also in the full NLO formula.

which has again the form of eq. (7.84) for a single variable $k_{\rm T}$ instead of the set of variables ξ, y. The integral in eq. (7.87) is still too complex to be performed analytically, so we resort a second time to the veto method. Using the inequality

$$\log \frac{4k_{\rm max}^2}{k_{\rm T}^2} \geqslant \log \frac{1+\sqrt{1-(k_{\rm T}/k_{\rm max})^2}}{1-\sqrt{1-(k_{\rm T}/k_{\rm max})^2}}\,, \tag{7.88}$$

we generate the $p_{\rm T}$ distribution according to $d\Delta_{uu}(p_{\rm T})$, where

$$\begin{aligned} \Delta_{uu}(p_{\rm T}) &= \exp\left[-2\pi N \int_0^{k_{\rm max}} \frac{dk_{\rm T}}{k_{\rm T}}\, \alpha_{\rm S}(k_{\rm T})\, \log \frac{4k_{\rm max}^2}{k_{\rm T}^2}\, \theta(k_{\rm T}-p_{\rm T})\right] \\ &= \exp\left[-\frac{\pi N}{b_0}\left(\log \frac{4k_{\rm max}^2}{\Lambda^2} \log \frac{\log\left(k_{\rm max}^2/\Lambda^2\right)}{\log\left(p_{\rm T}^2/\Lambda^2\right)} - \log \frac{k_{\rm max}^2}{p_{\rm T}^2}\right)\right]. \end{aligned} \tag{7.89}$$

Observe that we have a lower limit on the acceptable values of $p_{\rm T}$, since $\alpha_{\rm S}(p_{\rm T})$ must be well defined for both the two-loop and the one-loop expression of $\alpha_{\rm S}$. We thus introduce a $p_{\rm T}^{\rm min} \gtrsim \Lambda$. Summarizing

1. Set $p_{\rm T}^{\rm max} = k_{\rm max}$, the maximum allowed value according to eq. (7.86). We have $\Delta_{uu}(p_{\rm T}^{\rm max}) = 1$.

2. Generate a uniformly-distributed random number r between 0 and 1, and solve the equation $r = \Delta_{uu}(p_{\rm T})/\Delta_{uu}(p_{\rm T}^{\rm max})$ for $p_{\rm T}$. If no solution is found, or if $p_{\rm T} < p_{\rm T}^{\rm min}$, there is no radiation, and a $q\bar{q}$ event is generated.

3. If a solution with $p_{\rm T} > p_{\rm T}^{\rm min}$ is found, generate r' with $0 \leqslant r' \leqslant \log \frac{4k_{\rm max}^2}{p_{\rm T}^2}$. If $r' \leqslant \log \frac{1+\sqrt{1-(p_{\rm T}/k_{\rm max})^2}}{1-\sqrt{1-(p_{\rm T}/k_{\rm max})^2}}$ go to step 4. Otherwise set $p_{\rm T}^{\rm max} = p_{\rm T}$ and go to step 2.

4. At this point $p_{\rm T}$ is generated according to eq. (7.87). The radiation variables ϕ, y, ξ are then distributed with a probability proportional to

$$\frac{1}{\xi\,(1-y^2)} \delta(k_{\rm T}(\xi, y) - p_{\rm T})\, d\phi\, dy\, d\xi\,, \tag{7.90}$$

where $k_{\rm T}(\xi, y)$ is given by eq. (7.79). The δ function does not constraint the azimuthal angle ϕ, that is then generated uniformly between 0 and 2π, but it fixes ξ to be

$$\xi = \frac{2p_{\rm T}}{q^0\sqrt{1-y^2}}\,, \tag{7.91}$$

and, upon integrating in ξ, y is distributed with a probability proportional to

$$\theta\left(\sqrt{1-y^2} - \frac{2p_{\rm T}}{q^0}\right) d\log \frac{1+y}{1-y}\,. \tag{7.92}$$

One thus generates a uniform random number r_y between the minimum and maximum value of the logarithm allowed by the θ function

$$-\log \frac{1+\sqrt{1-(p_{\rm T}/k_{\rm max})^2}}{1-\sqrt{1-(p_{\rm T}/k_{\rm max})^2}} < r_y < \log \frac{1+\sqrt{1-(p_{\rm T}/k_{\rm max})^2}}{1-\sqrt{1-(p_{\rm T}/k_{\rm max})^2}} \tag{7.93}$$

and solves the equation

$$r_y = \log \frac{1+y}{1-y} \tag{7.94}$$

for y.

5. Now the last veto: generate a random number r'' between 0 and $U(\xi, y)$. If

$$r'' \leqslant \sum_{\alpha_r \in \{32,31\}} \left[\frac{J\, R(\bar{\theta}_1, \phi, \xi, y)}{B(\bar{\theta}_1)} \right]_{\alpha_r} \tag{7.95}$$

accept the event. Otherwise set $p_{\mathrm{T}}^{\max} = p_{\mathrm{T}}$ and go to step 2.

6. If

$$r'' \geqslant \left[\frac{J\, R(\bar{\theta}_1, \phi, \xi, y)}{B(\bar{\theta}_1)} \right]_{\alpha_r = 32}$$

set $\alpha_r = 31$. Otherwise set $\alpha_r = 32$.

This completes the generation of the radiation variables and of the singular region α_r. Observe that, in our simple case, there is total symmetry between the regions 31 and 32. One could simply double the contribution of region 32, and, at the end, choose one of the two regions with equal probability. The rather pedantic discussion given above has the only purpose of better clarifying the method in general.

7.3 $h_\oplus h_\ominus \to V$ in the Catani and Seymour formalism

Born term. In this section, we consider the production of a massive vector boson V of mass M in hadron collision, in the CS formalism,

$$q(\bar{k}_\oplus) + \bar{q}(\bar{k}_\ominus) \;\to\; V(\bar{k}_1)\,, \tag{7.96}$$

with $\bar{k}_1^2 = M^2$. Denoting, as usual, with S the centre-of-mass energy of the two incoming hadronic beams, $S = (K_\oplus + K_\ominus)^2$, and with $\bar{Y}$ the rapidity of the produced vector boson,

$$\bar{Y} = \frac{1}{2} \log \frac{\bar{k}_1^0 + \bar{k}_1^3}{\bar{k}_1^0 - \bar{k}_1^3}\,, \tag{7.97}$$

we have, from eq. (2.7),

$$d\bar{\mathbf{\Phi}}_1 \equiv d\bar{x}_\oplus\, d\bar{x}_\ominus\, \frac{d^3 \bar{k}_1}{(2\pi)^3 2\bar{k}_1^0}\, (2\pi)^4\, \delta^4\left(\bar{k}_\oplus + \bar{k}_\ominus - \bar{k}_1\right) = \frac{2\pi}{S} d\bar{Y}\,, \tag{7.98}$$

with the constraints

$$\bar{x}_\oplus = \sqrt{\frac{M^2}{S}} e^{\bar{Y}}, \qquad\qquad \bar{x}_\ominus = \sqrt{\frac{M^2}{S}} e^{-\bar{Y}}\,, \tag{7.99}$$

and momentum conservation implies

$$2\, \bar{k}_\oplus \cdot \bar{k}_\ominus = \bar{x}_\oplus \bar{x}_\ominus S = M^2\,. \tag{7.100}$$

The only Born variable is then $\bar{Y}$, so that

$$\bar{\mathbf{\Phi}}_1 = \left\{\bar{Y}\right\} . \tag{7.101}$$

The squared tree-level amplitude, averaged over color and polarization of incoming partons $(1/(4N_c^2))$, is given by

$$\overline{|\mathcal{M}_B(\bar{k}_\oplus, \bar{k}_\ominus)|^2} = \frac{2}{N_c}\, g^2 \left(\bar{k}_\oplus \cdot \bar{k}_\ominus\right) , \tag{7.102}$$

where g is the $Vq\bar{q}$ coupling. Introducing the flux factor $1/(2M^2)$, we obtain the differential cross section

$$\mathcal{B}_{q\bar{q}}\big(\bar{\mathbf{\Phi}}_1\big) \equiv \mathcal{B}_{q\bar{q}}\big(\bar{k}_\oplus, \bar{k}_\ominus\big) = \frac{1}{2N_c}\, g^2 , \tag{7.103}$$

independent from the external momenta. We denote by $q\bar{q}$ the Born flavour index f_b, that, in this example, carries information on the flavour of the (massless) incoming partons along the $\oplus$ and $\ominus$ direction. Using a standard convention for the numbering scheme, we have $q = -5, \ldots, -1, 1, \ldots, 5$, and $\bar{q} = -q$.

Virtual corrections. With the scale choice $Q^2 = M^2$, we have from eqs. (2.92) and (2.107)

$$\mathcal{V}_{\mathrm{fin},\, q\bar{q}}\big(\bar{\mathbf{\Phi}}_1\big) = C_{\mathrm{F}} \left(\pi^2 - 8\right) \mathcal{B}_{q\bar{q}}\big(\bar{\mathbf{\Phi}}_1\big) , \qquad \mathcal{V}_{q\bar{q}}\big(\bar{\mathbf{\Phi}}_1\big) = \frac{\alpha_{\mathrm{S}}}{\pi}\, C_{\mathrm{F}}\, \mathcal{B}_{q\bar{q}}\big(\bar{\mathbf{\Phi}}_1\big) . \tag{7.104}$$

Real corrections. Three different sub-processes contribute to the real radiation production

$$q(k_\oplus)\, \bar{q}(k_\ominus) \;\to\; V(k_1)\, g(k_2) , \tag{7.105}$$

$$q(k_\oplus)\, g(k_\ominus) \;\to\; V(k_1)\, q(k_2) , \tag{7.106}$$

$$g(k_\oplus)\, \bar{q}(k_\ominus) \;\to\; V(k_1)\, \bar{q}(k_2) . \tag{7.107}$$

Introducing the usual Mandelstam variables

$$s = (k_\oplus + k_\ominus)^2 = (k_1 + k_2)^2, \tag{7.108}$$

$$t = (k_\oplus - k_1)^2 = (k_\ominus - k_2)^2, \tag{7.109}$$

$$u = (k_\oplus - k_2)^2 = (k_\ominus - k_1)^2, \tag{7.110}$$

we can express them in terms of the two sets of radiation variables (see section 6.4): the $\oplus$ set, $\Phi_{\mathrm{rad}}^{\oplus} = \{x_{\oplus\ominus}, \tilde{v}_\oplus, \phi\}$, that describes the emission from the $\oplus$ parton, and the $\ominus$ set, $\Phi_{\mathrm{rad}}^{\ominus} = \{x_{\oplus\ominus}, \tilde{v}_\ominus, \phi\}$, that describes the emission from the $\ominus$ parton

- $\Phi_{\mathrm{rad}}^{\oplus}$

$$s = \frac{M^2}{x_{\oplus\ominus}}, \qquad t = -s\,(1 - x_{\oplus\ominus} - \tilde{v}_\oplus) , \qquad u = -s\, \tilde{v}_\oplus, \tag{7.111}$$

- $\Phi_{\mathrm{rad}}^{\ominus}$

$$s = \frac{M^2}{x_{\oplus\ominus}}, \qquad t = -s\, \tilde{v}_\ominus, \qquad u = -s\,(1 - x_{\oplus\ominus} - \tilde{v}_\ominus) , \tag{7.112}$$

where we have dropped the i subscript, since we have only one radiated parton, and we have put a $\oplus\!\!\!\!\ominus$ subscript on the angular $\tilde{v}$ variable, in order to distinguish the two sets.

The squared amplitudes corresponding to the processes in eqs. (7.105)–(7.107) are:

1. The squared amplitude for the $q(k_\oplus)\,\bar{q}(k_\ominus) \;\to\; V(k_1)\,g(k_2)$ process, averaged over color and polarization of incoming partons $(1/(4N_c^2))$, is given by

$$\overline{|\mathcal{M}_{q\bar{q}}|^2} = 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\left[\frac{u}{t}+\frac{t}{u}+\frac{2\,s\,M^2}{tu}\right] \tag{7.113}$$

$$= 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\left[\frac{\tilde{v}_{\oplus\!\!\!\ominus}}{1-x_{\oplus\ominus}-\tilde{v}_{\oplus\!\!\!\ominus}}+\frac{1-x_{\oplus\ominus}-\tilde{v}_{\oplus\!\!\!\ominus}}{\tilde{v}_{\oplus\!\!\!\ominus}}+\frac{2x_{\oplus\ominus}}{\tilde{v}_{\oplus\!\!\!\ominus}\,(1-x_{\oplus\ominus}-\tilde{v}_{\oplus\!\!\!\ominus})}\right]. \tag{7.114}$$

   This expression has singularities when the gluon is emitted both along the $\oplus$ and the $\ominus$ direction ($\tilde{v}_{\oplus\!\!\!\ominus}\to 0$).

2. The squared amplitude for the $q(k_\oplus)\,g(k_\ominus) \;\to\; V(k_1)\,q(k_2)$ process, averaged over color and polarization of incoming partons $(1/(4N_c(N_c^2-1)))$, is equal to

$$\overline{|\mathcal{M}_{qg}|^2} = -2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\left[\frac{s}{t}+\frac{t}{s}+\frac{2\,u\,M^2}{st}\right] \tag{7.115}$$

$$= 2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{\tilde{v}_\ominus}\left[\tilde{v}_\ominus^2+1-2x_{\oplus\ominus}\,(1-x_{\oplus\ominus}-\tilde{v}_\ominus)\right]. \tag{7.116}$$

3. The squared amplitude for the $g(k_\oplus)\,\bar{q}(k_\ominus) \;\to\; V(k_1)\,\bar{q}(k_2)$ process, averaged over color and polarization of incoming partons, is given by

$$\overline{|\mathcal{M}_{g\bar{q}}|^2} = -2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\left[\frac{u}{s}+\frac{s}{u}+\frac{2\,t\,M^2}{su}\right] \tag{7.117}$$

$$= 2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{\tilde{v}_\oplus}\left[\tilde{v}_\oplus^2+1-2x_{\oplus\ominus}\,(1-x_{\oplus\ominus}-\tilde{v}_\oplus)\right]. \tag{7.118}$$

Counterterms. According to eqs. (5.136), (5.145) and (5.146) of ref. [28], the counterterms are given by

$$\mathcal{D}^{qg,\bar{q}} = -\frac{1}{u}\frac{1}{x_{\oplus\ominus}}2\,g_s^2\,C_{\rm F}\left[\frac{2}{1-x_{\oplus\ominus}}-(1+x_{\oplus\ominus})\right]\overline{|\mathcal{M}_B(x_{\oplus\ominus}\,k_\oplus,k_\ominus)|^2}$$

$$= 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{\tilde{v}_\oplus}\left[\frac{2}{1-x_{\oplus\ominus}}-(1+x_{\oplus\ominus})\right] \tag{7.119}$$

$$\mathcal{D}^{\bar{q}g,q} = -\frac{1}{t}\frac{1}{x_{\oplus\ominus}}2\,g_s^2\,C_{\rm F}\left[\frac{2}{1-x_{\oplus\ominus}}-(1+x_{\oplus\ominus})\right]\overline{|\mathcal{M}_B(k_\oplus,x_{\oplus\ominus}\,k_\ominus)|^2}$$

$$= 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{\tilde{v}_\ominus}\left[\frac{2}{1-x_{\oplus\ominus}}-(1+x_{\oplus\ominus})\right] \tag{7.120}$$

for gluon radiation from a $q\bar{q}$ initial-state, and

$$\mathcal{D}^{g\bar{q},\bar{q}} = -\frac{1}{u}\frac{1}{x_{\oplus\ominus}}2\,g_s^2\,T_{\rm F}\left[1-2\,x_{\oplus\ominus}\,(1-x_{\oplus\ominus})\right]\overline{|\mathcal{M}_B(x_{\oplus\ominus}\,k_\oplus,k_\ominus)|^2}$$

$$= 2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{\tilde{v}_\oplus}\left[1-2\,x_{\oplus\ominus}\,(1-x_{\oplus\ominus})\right] \tag{7.121}$$

$$\mathcal{D}^{qg,q} = -\frac{1}{t}\frac{1}{x_{\oplus\ominus}} 2\,g_s^2\,T_{\rm F}\left[1-2\,x_{\oplus\ominus}\,(1-x_{\oplus\ominus})\right]\overline{|\mathcal{M}_B(k_\oplus, x_{\oplus\ominus}\,k_\ominus)|^2}$$

$$= 2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{\tilde{v}_\ominus}\left[1-2\,x_{\oplus\ominus}\,(1-x_{\oplus\ominus})\right] \tag{7.122}$$

for the $g\bar{q}$ and qg initiated processes, respectively.

Radiation variables and inverse construction. Following section 6.4.2, given the rapidity $\bar{Y}$, the invariant squared mass of the produced boson M^2 and the three radiation variables ($x_{\oplus\ominus}$, $\tilde{v}_{\oplus\!\!\!\!\ominus}$ and ϕ), we can construct the full 2-body event.

For definiteness, we assume that the emitting parton is the $\oplus$ one. A similar procedure can be followed if the emitting parton is the $\ominus$ one, with the exchange of the roles of the $\oplus$ and $\ominus$ directions. Using eq. (7.99), we can compute $\bar{x}_\oplus$ and $\bar{x}_\ominus$, and from eqs. (6.91)–(6.93), we have

$$k_\oplus = x_\oplus K_\oplus = \frac{\bar{x}_\oplus}{x_{\oplus\ominus}} K_\oplus\,, \qquad\qquad k_\ominus = x_\ominus K_\ominus = \bar{x}_\ominus K_\ominus\,. \tag{7.123}$$

In the centre-of-mass system of $(k_\oplus + k_\ominus)$, the energy $k_2'^0$ of the final-state parton and its angle θ_2' with respect to the $k_\oplus$ direction are given by (see eqs. (6.96) and (6.97))

$$k_2'^0 = \sqrt{\frac{k_\oplus\cdot k_\ominus}{2}}\,(1-x_{\oplus\ominus})\,, \qquad \cos\theta_2' = \frac{1-2\tilde{v}_\oplus - x_{\oplus\ominus}}{1-x_{\oplus\ominus}}\,, \tag{7.124}$$

so that

$$k_2'^1 = k_2'^0\,\sin\theta_2'\,\cos\phi\,, \qquad k_2'^2 = k_2'^0\,\sin\theta_2'\,\sin\phi\,, \qquad k_2'^3 = k_2'^0\,\cos\theta_2'\,. \tag{7.125}$$

We can now boost back in the laboratory frame the momentum k_2', with a boost parallel to the $\oplus$ direction, with velocity given by

$$\beta = \frac{x_\oplus - x_\ominus}{x_\oplus + x_\ominus}\,, \tag{7.126}$$

and obtain the radiated parton momentum k_2. There is no need of further Lorentz boosts to obtain k_1, since we can simply use momentum conservation

$$k_1 = k_\oplus + k_\ominus - k_2\,. \tag{7.127}$$

According to eq. (6.90), the infinitesimal phase-space volume, $d\mathbf{\Phi}_2$, can be written as

$$dx_\oplus\,dx_\ominus\,d\Phi_2 = d\bar{\mathbf{\Phi}}_1\,d\Phi_{\rm rad}^\oplus \tag{7.128}$$

where

$$d\Phi_{\rm rad}^\oplus = \frac{(2k_\oplus\cdot k_\ominus)}{16\pi^2}\,\frac{d\phi}{2\pi}\,d\tilde{v}_\oplus\,\frac{dx_{\oplus\ominus}}{x_{\oplus\ominus}}\,\theta(\tilde{v}_\oplus)\,\theta\!\left(1-\frac{\tilde{v}_\oplus}{1-x_{\oplus\ominus}}\right)\theta\big(x_{\oplus\ominus}(1-x_{\oplus\ominus})\big)\,\theta(x_{\oplus\ominus}-\bar{x}_\oplus)\,. \tag{7.129}$$

The expression for the infinitesimal phase-space radiation volume, in the case of emission from the $\ominus$ parton, $d\Phi_{\rm rad}^\ominus$, can be obtained with the interchange $\tilde{v}_\oplus \leftrightarrow \tilde{v}_\ominus$, $\bar{x}_\oplus \leftrightarrow \bar{x}_\ominus$. The θ functions appearing in eq. (7.129) have a simple physical interpretation: they constrain the energy in eq. (7.124) to be positive, the cosine of the θ_2' angle (see eq. (7.124)) to be in the appropriate range and force $x_\oplus$ and $x_\ominus$ to be in the physical range.

Collinear remnants. The collinear remnants of eq. (2.108) are given by

$$\mathcal{G}_{\oplus}^{q\bar{q}}(\bar{\mathbf{\Phi}}_{1,\oplus}) = \frac{\alpha_{\rm S}}{2\pi}C_{\rm F}\Bigg[\left(\frac{2}{1-z}\log\frac{(1-z)^2}{z}\right)_+ - (1+z)\log\frac{(1-z)^2}{z} + (1-z) + \left(\frac{2}{3}\pi^2 - 5\right)\delta(1-z) + \left(\frac{1+z^2}{1-z}\right)_+ \log\frac{M^2}{\mu_{\rm F}^2}\Bigg]\mathcal{B}_{q\bar{q}}(zk_\oplus, k_\ominus)\,, \tag{7.130}$$

$$\mathcal{G}_{\oplus}^{g\bar{q}}(\bar{\mathbf{\Phi}}_{1,\oplus}) = \frac{\alpha_{\rm S}}{2\pi}T_{\rm F}\left\{\left[z^2 + (1-z)^2\right]\left[\log\frac{(1-z)^2}{z} + \log\frac{M^2}{\mu_{\rm F}^2}\right] + 2z(1-z)\right\}\mathcal{B}_{q\bar{q}}(zk_\oplus, k_\ominus)\,. \tag{7.131}$$

The other two collinear remnants $\mathcal{G}_{\ominus}^{q\bar{q}}(\bar{\mathbf{\Phi}}_{1,\ominus})$ and $\mathcal{G}_{\ominus}^{qg}(\bar{\mathbf{\Phi}}_{1,\ominus})$ are equal to $\mathcal{G}_{\oplus}^{q\bar{q}}(\bar{\mathbf{\Phi}}_{1,\oplus})$ and $\mathcal{G}_{\oplus}^{g\bar{q}}(\bar{\mathbf{\Phi}}_{1,\oplus})$ respectively, with $\mathcal{B}_{q\bar{q}}(zk_\oplus, k_\ominus)$ replaced by $\mathcal{B}_{q\bar{q}}(k_\oplus, zk_\ominus)$.

POWHEG ingredients. We define the contributions to the real differential cross section, and corresponding counterterms, according to the three singular regions, by attaching the flux factor $1/(2s) = x_{\oplus\ominus}/(2M^2)$, to the squared matrix elements

$$\mathcal{R}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\overline{|\mathcal{M}_{q\bar{q}}|^2}\,, \tag{7.132}$$

$$\mathcal{R}_{qg}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\ominus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\overline{|\mathcal{M}_{qg}|^2}\,, \tag{7.133}$$

$$\mathcal{R}_{g\bar{q}}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\overline{|\mathcal{M}_{g\bar{q}}|^2}\,, \tag{7.134}$$

and

$$\mathcal{C}_{q\bar{q}}^{\oplus}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\mathcal{D}^{qg,\bar{q}}\,, \tag{7.135}$$

$$\mathcal{C}_{q\bar{q}}^{\ominus}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\ominus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\mathcal{D}^{qg,q}\,, \tag{7.136}$$

$$\mathcal{C}_{g\bar{q}}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\mathcal{D}^{qg,q}\,, \tag{7.137}$$

$$\mathcal{C}_{qg}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\ominus}) = \frac{x_{\oplus\ominus}}{2M^2}\,\mathcal{D}^{g\bar{q},\bar{q}}\,. \tag{7.138}$$

We also define

$$\mathcal{R}_{q\bar{q}}^{\oplus}\left(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}\right) = \mathcal{R}_{q\bar{q}}\left(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}\right)\left[\frac{\mathcal{C}_{q\bar{q}}^{\oplus}}{\mathcal{C}_{q\bar{q}}^{\oplus} + \mathcal{C}_{q\bar{q}}^{\ominus}}\right]_{\Phi_{\rm rad}^{\oplus}}\,, \tag{7.139}$$

$$\mathcal{R}_{q\bar{q}}^{\ominus}\left(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\ominus}\right) = \mathcal{R}_{q\bar{q}}\left(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\ominus}\right)\left[\frac{\mathcal{C}_{q\bar{q}}^{\ominus}}{\mathcal{C}_{q\bar{q}}^{\oplus} + \mathcal{C}_{q\bar{q}}^{\ominus}}\right]_{\Phi_{\rm rad}^{\ominus}}\,, \tag{7.140}$$

where the counterterms are evaluated at the values specified by the radiation variables.

The expression for $\bar{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)$ of eq. (4.13) is then given by

$$\bar{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) = B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) + V_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) + \int d\Phi_{\rm rad}^{\oplus}\left[R_{q\bar{q}}^{\oplus}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus}) - C_{q\bar{q}}^{\oplus}(\bar{\mathbf{\Phi}}_1, \Phi_{\rm rad}^{\oplus})\right] \tag{7.141}$$

$$+\int d\Phi^{\ominus}_{\rm rad}\left[R^{\ominus}_{q\bar q}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad})-C^{\ominus}_{q\bar q}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad})\right]$$
$$+\int d\Phi^{\oplus}_{\rm rad}\left[R_{g\bar q}(\bar{\mathbf{\Phi}}_1,\Phi^{\oplus}_{\rm rad})-C_{g\bar q}(\bar{\mathbf{\Phi}}_1,\Phi^{\oplus}_{\rm rad})\right]$$
$$+\int d\Phi^{\ominus}_{\rm rad}\left[R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad})-C_{qg}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad})\right]$$
$$+\int_{\bar x_\oplus}^1\frac{dz}{z}\left[G^{q\bar q}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})+G^{g\bar q}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})\right]+\int_{\bar x_\ominus}^1\frac{dz}{z}\left[G^{q\bar q}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})+G^{qg}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})\right],$$

where

$$B_{q\bar q}(\bar{\mathbf{\Phi}}_1)=\mathcal{B}_{q\bar q}(\bar{\mathbf{\Phi}}_1)\;\mathcal{L}_{q\bar q}(\bar x_\oplus,\bar x_\ominus)\,,\qquad V_{q\bar q}(\bar{\mathbf{\Phi}}_1)=\mathcal{V}_{q\bar q}(\bar{\mathbf{\Phi}}_1)\;\mathcal{L}_{q\bar q}(\bar x_\oplus,\bar x_\ominus)\,,\tag{7.142}$$

and the quantities R and C are obtained multiplying the corresponding quantities $\mathcal{R}$ and $\mathcal{C}$ by the corresponding factor $\mathcal{L}(x_\oplus,x_\ominus)$. Furthermore

$$G^{q\bar q}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})=\mathcal{G}^{q\bar q}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})\,\mathcal{L}_{q\bar q}\left(\frac{\bar x_\oplus}{z},\bar x_\ominus\right),\tag{7.143}$$
$$G^{g\bar q}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})=\mathcal{G}^{g\bar q}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})\,\mathcal{L}_{g\bar q}\left(\frac{\bar x_\oplus}{z},\bar x_\ominus\right),\tag{7.144}$$
$$G^{q\bar q}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})=\mathcal{G}^{q\bar q}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})\,\mathcal{L}_{q\bar q}\left(\bar x_\oplus,\frac{\bar x_\ominus}{z}\right),\tag{7.145}$$
$$G^{qg}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})=\mathcal{G}^{qg}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})\,\mathcal{L}_{qg}\left(\bar x_\oplus,\frac{\bar x_\ominus}{z}\right),\tag{7.146}$$

with

$$\mathcal{L}_{ff'}(x_\oplus,x_\ominus)=f^{\oplus}_f(x_\oplus,\mu_{\rm F}^2)\;f^{\ominus}_{f'}(x_\ominus,\mu_{\rm F}^2)\,.\tag{7.147}$$

where $f^f_{\oplus\!\!\!\ominus}(x,\mu_{\rm F}^2)$ is the parton density function of the parton f in the hadron $\oplus\!\!\!\ominus$, and $\mu_{\rm F}^2$ is the factorization scale.

Generation of the Born variable $\bar{\boldsymbol{Y}}$. From eq. (7.129), we can write

$$d\Phi^{\oplus\!\!\!\ominus}_{\rm rad}=\frac{M^2}{16\pi^2}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_{\bar x_{\oplus\!\!\!\ominus}}^1 dx_{\oplus\ominus}\int_0^{1-x_{\oplus\ominus}}d\tilde v_{\oplus\!\!\!\ominus}\,\frac{1}{(x_{\oplus\ominus})^2}\,,\tag{7.148}$$

and with the change of variables

$$\phi=2\,\pi\,\varphi\,,\qquad \tilde v_{\oplus\!\!\!\ominus}=(1-x_{\oplus\ominus})\,\tilde v\,,\qquad x_{\oplus\ominus}=\bar x_{\oplus\!\!\!\ominus}+(1-\bar x_{\oplus\!\!\!\ominus})\,x\,,\tag{7.149}$$

we have

$$d\Phi^{\oplus\!\!\!\ominus}_{\rm rad}=\frac{M^2}{16\pi^2}(1-\bar x_{\oplus\!\!\!\ominus})^2\int_0^1 d\varphi\int_0^1 dx\,(1-x)\int_0^1 d\tilde v\;\frac{1}{(\bar x_{\oplus\!\!\!\ominus}+(1-\bar x_{\oplus\!\!\!\ominus})\,x)^2}\,,\tag{7.150}$$

so that now both $d\Phi^{\oplus}_{\rm rad}$ and $d\Phi^{\ominus}_{\rm rad}$ are expressed in terms of the same integration variables and integration ranges.

The set of variables $\{\varphi,\,x,\,\tilde v\}$ are equivalent to the three $X_{\rm rad}$ variables, introduced in section 4.4.1. We can now insert the process-dependent part of the Jacobian into the integrand, and define $d^3X_{\rm rad}$ as

$$\int d^3X_{\rm rad}=\int_0^1 d\varphi\int_0^1 dx\int_0^1 d\tilde v\,,\tag{7.151}$$

so that

$$d\Phi^{\oplus\!\!\!\!\ominus}_{\rm rad} = d^3X_{\rm rad}\, J^{\oplus\!\!\!\!\ominus}\,, \qquad\qquad J^{\oplus\!\!\!\!\ominus} = \frac{M^2}{16\pi^2}\,(1-\bar{x}_{\oplus\!\!\!\!\ominus})^2\,(1-x)\frac{1}{(\bar{x}_{\oplus\!\!\!\!\ominus}+(1-\bar{x}_{\oplus\!\!\!\!\ominus})\,x)^2}\,. \tag{7.152}$$

In addition, we make a change of variable in the integrals of the two collinear remnants of eq. (7.141),

$$z = \bar{x}_{\oplus\!\!\!\!\ominus} + (1-\bar{x}_{\oplus\!\!\!\!\ominus})\,x, \qquad \text{and} \qquad \frac{dz}{dx} = 1-\bar{x}_{\oplus\!\!\!\!\ominus}\,, \tag{7.153}$$

so that the limits of x are consistently between 0 and 1. The $\tilde{B}$ function of eq. (4.24) is then given by

$$\begin{aligned}\tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, X_{\rm rad}) \;=\; & B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) + V_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) \\ & + J^{\oplus}\left[R^{\oplus}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi^{\oplus}_{\rm rad}) - C^{\oplus}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi^{\oplus}_{\rm rad})\right] + J^{\ominus}\left[R^{\ominus}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad}) - C^{\ominus}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad})\right] \\ & + J^{\oplus}\left[R_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi^{\oplus}_{\rm rad}) - C_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi^{\oplus}_{\rm rad})\right] + J^{\ominus}\left[R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad}) - C_{qg}(\bar{\mathbf{\Phi}}_1,\Phi^{\ominus}_{\rm rad})\right] \\ & + \frac{1-\bar{x}_{\oplus}}{z}\left[G^{q\bar{q}}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus}) + G^{g\bar{q}}_{\oplus}(\bar{\mathbf{\Phi}}_{1,\oplus})\right] + \frac{1-\bar{x}_{\ominus}}{z}\left[G^{q\bar{q}}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus}) + G^{qg}_{\ominus}(\bar{\mathbf{\Phi}}_{1,\ominus})\right],\end{aligned} \tag{7.154}$$

so that

$$\int d^3X_{\rm rad}\,\tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, X_{\rm rad}) = \bar{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)\,. \tag{7.155}$$

The generation of the Born variable $\bar{Y}$ is performed using the method illustrated in section 4.4.1. We use an integrator-unweighter program (like for example the `BASES-SPRING` package), that, after a single 4-dimensional integration of the function

$$\tilde{B}(\bar{\mathbf{\Phi}}_1, X_{\rm rad}) = \sum_q \tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, X_{\rm rad})\,, \tag{7.156}$$

can generate 4-tuples of $\{\bar{Y},\,\varphi,\,x,\,\tilde{v}\}$ values, distributed according to $\tilde{B}(\bar{\mathbf{\Phi}}_1, X_{\rm rad})$. For each generated 4-tuple, we generate q with a probability proportional to $\tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, X_{\rm rad})$.[19] We only keep the $\bar{Y}$ and q generated value, and neglect φ, x, $\tilde{v}$, which corresponds to integrate over them.

Observe that $\tilde{B}(\bar{\mathbf{\Phi}}_1, X_{\rm rad})$ is positive, unless the $\mathcal{O}(\alpha_{\rm S})$ terms overcome the Born term. If this happens, the whole perturbative approach breaks down. This can happen if M is too small, or if we are in extreme regions of phase space, like the threshold region (i.e. when the values of $\bar{Y}$ and M are forcing $\bar{x}_{\oplus}$ or $\bar{x}_{\ominus}$ to 1). In practical applications, it is wise to check that the fraction of negative weights in the integral of $\tilde{B}(\bar{\mathbf{\Phi}}_1, X_{\rm rad})$ is small, and can be safely neglected.

Generation of the radiation variables. The transverse momentum $k_{\rm T}$ of the radiation, with respect to the incoming beams, in the two singular regions is given by

$$k^2_{{\rm T}\oplus\!\!\!\!\ominus} = s\,(1-x_{\oplus\ominus})\,\tilde{v}_{\oplus\!\!\!\!\ominus} = M^2\,\frac{1-x_{\oplus\ominus}}{x_{\oplus\ominus}}\,\tilde{v}_{\oplus\!\!\!\!\ominus}\,, \tag{7.157}$$

[19] The standard procedure to generate an index $0 < j \leqslant n$ with a probability proportional to a_j is to generate a random number $0 < r < 1$, and then choose j so that $\sum_{j'=1}^{j-1} a_{j'} < r\sum_{j'=1}^{n} a_{j'} < \sum_{j'=1}^{j} a_{j'}$.

where $\tilde{v}_\oplus$ is used when the emitter is the $\oplus$ parton, and $\tilde{v}_\ominus$ when it is the $\ominus$ one, and the Sudakov form factor of eq. (4.16) is

$$\Delta_q\left(\bar{\mathbf{\Phi}}_1, p_{\rm T}\right) = \exp\left\{-\int d\Phi_{\rm rad}^{\oplus}\frac{R_{q\bar{q}}^{\oplus}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\oplus})+R_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\oplus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)}\,\theta\big(k_{{\rm T}\oplus}^2-p_{\rm T}^2\big)\right. \tag{7.158}$$

$$\left.-\int d\Phi_{\rm rad}^{\ominus}\frac{R_{q\bar{q}}^{\ominus}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\ominus})+R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\ominus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)}\,\theta\big(k_{{\rm T}\ominus}^2-p_{\rm T}^2\big)\right\}.$$

In order to generate the radiation variables, we use the hit-and-miss technique introduced in section 4.4.2. Suitable upper bounding functions are given by

$$\frac{R_{q\bar{q}}^{\oplus}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\oplus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \lesssim 8\,C_{\rm F}\,g_s^2\,\frac{x_{\oplus\ominus}}{2M^2}\,\frac{1}{\tilde{v}_\oplus}\,\frac{2}{1-x_{\oplus\ominus}}\,\frac{f_\oplus^{q}(\bar{x}_\oplus/x_{\oplus\ominus},\mu_{\rm F}^2)}{f_\oplus^{q}(\bar{x}_\oplus,\mu_{\rm F}^2)}\,, \tag{7.159}$$

$$\frac{R_{q\bar{q}}^{\ominus}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\ominus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \lesssim 8\,C_{\rm F}\,g_s^2\,\frac{x_{\oplus\ominus}}{2M^2}\,\frac{1}{\tilde{v}_\ominus}\,\frac{2}{1-x_{\oplus\ominus}}\,\frac{f_\ominus^{\bar{q}}(\bar{x}_\ominus/x_{\oplus\ominus},\mu_{\rm F}^2)}{f_\ominus^{\bar{q}}(\bar{x}_\ominus,\mu_{\rm F}^2)}\,, \tag{7.160}$$

$$\frac{R_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\oplus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \lesssim 8\,T_{\rm F}\,g_s^2\,\frac{x_{\oplus\ominus}}{2M^2}\,\frac{1}{\tilde{v}_\oplus}\,\frac{f_\oplus^{g}(\bar{x}_\oplus/x_{\oplus\ominus},\mu_{\rm F}^2)}{f_\oplus^{q}(\bar{x}_\oplus,\mu_{\rm F}^2)}\,, \tag{7.161}$$

$$\frac{R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\ominus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \lesssim 8\,T_{\rm F}\,g_s^2\,\frac{x_{\oplus\ominus}}{2M^2}\,\frac{1}{\tilde{v}_\ominus}\,\frac{f_\ominus^{g}(\bar{x}_\ominus/x_{\oplus\ominus},\mu_{\rm F}^2)}{f_\ominus^{\bar{q}}(\bar{x}_\ominus,\mu_{\rm F}^2)}\,. \tag{7.162}$$

Following appendix D, we can put an upper bound on the ratios of the parton distribution functions (pdf) too, and we obtain

$$\frac{R_{q\bar{q}}^{\oplus}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\oplus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \le N_{q\bar{q}}^{\oplus}\,\frac{\alpha_{\rm S}}{\tilde{v}_\oplus}\,\frac{x_{\oplus\ominus}}{1-x_{\oplus\ominus}}\,, \tag{7.163}$$

$$\frac{R_{q\bar{q}}^{\ominus}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\ominus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \le N_{q\bar{q}}^{\ominus}\,\frac{\alpha_{\rm S}}{\tilde{v}_\ominus}\,\frac{x_{\oplus\ominus}}{1-x_{\oplus\ominus}}\,, \tag{7.164}$$

$$\frac{R_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\oplus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \le N_{g\bar{q}}\,\frac{\alpha_{\rm S}}{\tilde{v}_\oplus}\,\frac{x_{\oplus\ominus}}{1-x_{\oplus\ominus}}\,, \tag{7.165}$$

$$\frac{R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}^{\ominus})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)} \le N_{qg}\,\frac{\alpha_{\rm S}}{\tilde{v}_\ominus}\,\frac{x_{\oplus\ominus}}{1-x_{\oplus\ominus}}\,, \tag{7.166}$$

where $N_{q\bar{q}}^{\oplus\atop\ominus}$, $N_{g\bar{q}}$ and N_{qg} are upper-bound constants determined by sampling the whole phase-space of the R/B ratios.

We can then write an upper bound for the integrand function in the Sudakov form factor of eq. (7.158)

$$\int d\Phi_{\rm rad}^{\oplus}\frac{R_{q\bar{q}}^{\oplus}+R_{g\bar{q}}}{B_{q\bar{q}}}\,\theta\big(k_{{\rm T}\oplus}^2-p_{\rm T}^2\big)+\int d\Phi_{\rm rad}^{\ominus}\frac{R_{q\bar{q}}^{\ominus}+R_{qg}}{B_{q\bar{q}}}\,\theta\big(k_{{\rm T}\ominus}^2-p_{\rm T}^2\big)$$
$$\le N_\oplus\int_{\bar{x}_\oplus}^1 dx_{\oplus\ominus}\int_0^{1-x_{\oplus\ominus}} d\tilde{v}_\oplus\,\frac{1}{x_{\oplus\ominus}\,(1-x_{\oplus\ominus})}\,\frac{1}{\tilde{v}_\oplus}\,\alpha_{\rm S}\left(k_{{\rm T}\oplus}^2\right)\,\theta\big(k_{{\rm T}\oplus}^2-p_{\rm T}^2\big)$$
$$+N_\ominus\int_{\bar{x}_\ominus}^1 dx_{\oplus\ominus}\int_0^{1-x_{\oplus\ominus}} d\tilde{v}_\ominus\,\frac{1}{x_{\oplus\ominus}\,(1-x_{\oplus\ominus})}\,\frac{1}{\tilde{v}_\ominus}\,\alpha_{\rm S}\left(k_{{\rm T}\ominus}^2\right)\,\theta\big(k_{{\rm T}\ominus}^2-p_{\rm T}^2\big)\,, \tag{7.167}$$

where $N_{\oplus}$ are constant with respect to the radiation variables. The two integrals have the same functional form, so that we just need to generate radiation variables using the highest-$p_{\rm T}$ bid method of appendix B and the veto technique of appendix A, where the $p_{\rm T}$ is generated uniformly in $\Delta_u(p_{\rm T})$

$$\Delta_u(p_{\rm T}) = \exp\left\{-N\int_{\bar{x}}^{1} dx \int_0^{1-x} dv\, \frac{1}{x\,(1-x)}\frac{1}{v}\,\alpha_{\rm S}\left(k_{\rm T}^2\right)\,\theta\!\left(k_{\rm T}^2 - p_{\rm T}^2\right)\right\}, \tag{7.168}$$

where $k_{\rm T}$ must be of the order of the radiation transverse momentum in the collinear limit, and it must coincide with it in the soft-collinear limit. A suitable definition is given by

$$k_{\rm T}^2 = M^2\,(1-x)\,v\,. \tag{7.169}$$

Introducing the dimensionless quantities

$$\tilde{k}_{\rm T}^2 = \frac{k_{\rm T}^2}{M^2} = (1-x)\,v, \qquad\qquad \tilde{p}_{\rm T}^2 = \frac{p_{\rm T}^2}{M^2}\,, \tag{7.170}$$

and the integration over $d\tilde{k}_{\rm T}^2$, we have

$$\begin{aligned}
\Delta_u(p_{\rm T}) &= \exp\left\{-N\int_{\tilde{p}_{\rm T}^2}^{\tilde{k}_{\rm T,\,max}^2} d\tilde{k}_{\rm T}^2 \int_{\bar{x}}^{1} dx \int_0^{1-x} dv\, \frac{1}{x\,(1-x)}\frac{1}{v}\,\delta\!\left(v(1-x)-\tilde{p}_{\rm T}^2\right)\alpha_{\rm S}\left(k_{\rm T}^2\right)\right\}\\
&= \exp\left\{-N\int_{\tilde{p}_{\rm T}^2}^{\tilde{k}_{\rm T,\,max}^2} d\tilde{k}_{\rm T}^2 \int_{\bar{x}}^{1-\tilde{k}_{\rm T}} dx\, \frac{1}{x\,(1-x)}\,\frac{\alpha_{\rm S}\left(k_{\rm T}^2\right)}{\tilde{k}_{\rm T}^2}\,,\right\}\\
&= \exp\left\{-N\int_{\tilde{p}_{\rm T}^2}^{\tilde{k}_{\rm T,\,max}^2} d\tilde{k}_{\rm T}^2 \frac{\alpha_{\rm S}\left(k_{\rm T}^2\right)}{\tilde{k}_{\rm T}^2}\left[\log\frac{1-\tilde{k}_{\rm T}}{\tilde{k}_{\rm T}} + \log\frac{1-\bar{x}}{\bar{x}}\right]\right\},
\end{aligned} \tag{7.171}$$

where

$$\tilde{k}_{\rm T,\,max}^2 = (1-\bar{x})^2\,, \tag{7.172}$$

and $\alpha_{\rm S}\left(k_{\rm T}^2\right)$ is given by eq. (7.45). Since the generation of $p_{\rm T}$ according to

$$d\Delta_u(p_{\rm T}) = d\exp\left\{-N\int_0^{\tilde{k}_{\rm T,\,max}^2} d\tilde{k}_{\rm T}^2 \frac{\alpha_{\rm S}\left(k_{\rm T}^2\right)}{\tilde{k}_{\rm T}^2}\left[\log\frac{1-\tilde{k}_{\rm T}}{\tilde{k}_{\rm T}} + \log\frac{1-\bar{x}}{\bar{x}}\right]\theta\!\left(k_{\rm T}^2 - p_{\rm T}^2\right)\right\} \tag{7.173}$$

is still too complex to be performed analytically, we use a second time the veto method, using the upper bounding function

$$\log\frac{1}{\tilde{k}_{\rm T}} \geq \log\frac{1-\tilde{k}_{\rm T}}{\tilde{k}_{\rm T}}\,, \tag{7.174}$$

and we then generate the $p_{\rm T}$ distribution according to $d\Delta_{uu}\,(p_{\rm T})$, where

$$\begin{aligned}
\Delta_{uu}\,(p_{\rm T}) &= \exp\left\{-N\int_0^{\tilde{k}_{\rm T,\,max}^2} d\tilde{k}_{\rm T}^2\,\frac{\alpha_{\rm S}\left(k_{\rm T}^2\right)}{\tilde{k}_{\rm T}^2}\left[\log\frac{1}{\tilde{k}_{\rm T}} + \log\frac{1-\bar{x}}{\bar{x}}\right]\theta\!\left(k_{\rm T}^2 - p_{\rm T}^2\right)\right\}\\
&= \exp\left\{-\frac{N}{b_0}\left[-\frac{1}{2}\log\frac{M^2\,(1-\bar{x})^2}{p_{\rm T}^2}\right.\right.
\end{aligned}$$

$$+\log\frac{\log\left((1-\bar{x})^2 M^2/\Lambda^2\right)}{\log\left(p_{\rm T}^2/\Lambda^2\right)}\left(\log\frac{1-\bar{x}}{\bar{x}}+\frac{1}{2}\log\frac{M^2}{\Lambda^2}\right)\Bigg]\Bigg\}. \tag{7.175}$$

In order to apply the highest-$p_{\rm T}$ bid technique to eq. (7.158), we need to repeat the following steps for both the last two integrals in eq. (7.167). We follow in detail the first one. The second integral is treated in the same way.

Observe that we have a lower limit on the acceptable values of $p_{\rm T}$, since $\alpha_{\rm S}(p_{\rm T})$ must be well defined for both the two-loop and the one-loop expression of $\alpha_{\rm S}$. We thus introduce a $p_{\rm T}^{\rm min} \gtrsim \Lambda$.

When dealing with the first integral, we set $N = N_\oplus$ and $\bar{x} = \bar{x}_\oplus$.

1. Set $p_{\rm T}^{\rm max} = M\,(1-\bar{x})$, where $M^2\,(1-\bar{x})^2$ is the maximum value of $p_{\rm T}^2$, such that $\Delta_{uu}(p_{\rm T}^{\rm max}) = 1$.

2. Generate r with $0 < r < 1$, and solve the equation $r = \Delta_{uu}(p_{\rm T})/\Delta_{uu}(p_{\rm T}^{\rm max})$ for $p_{\rm T}$.

   If no solution is found, or if $p_{\rm T} < p_{\rm T}^{\rm min}$, no radiation is produced, and a Born event is generated.

3. If a solution with $p_{\rm T} > p_{\rm T}^{\rm min}$ is found, generate r' with $0 \leqslant r' \leqslant \log(M/p_{\rm T})+\log(1/\bar{x}-1)$. If $r' \leqslant \log(M/p_{\rm T}-1)+\log(1/\bar{x}-1)$ go to step 4. Otherwise set $p_{\rm T}^{\rm max} = p_{\rm T}$ and go to step 2.

4. At this point $p_{\rm T}$ is generated according to eq. (7.173). We need to generate x, v and ϕ with a probability proportional to

$$\frac{1}{x\,(1-x)}\,\frac{1}{v}\,\delta\big(M^2 v(1-x)-p_{\rm T}^2\big)dv\,dx\,d\phi. \tag{7.176}$$

   This is done as follows: integrate the above in v using the δ function, so that

$$v = \frac{p_{\rm T}^2}{M^2}\,\frac{1}{1-x}\,, \tag{7.177}$$

   and generate x according to

$$\theta(x-\bar{x})\;\theta\Big(1-\frac{p_{\rm T}}{M}-x\Big)\;d\log\frac{x}{1-x}\,, \tag{7.178}$$

   i.e., generate a uniformly-distributed random number r'' with

$$\log\frac{\bar{x}}{1-\bar{x}} \leqslant r'' \leqslant \log\left(\frac{M}{p_{\rm T}}-1\right) \tag{7.179}$$

   and solve $r'' = \log\big(x/(1-x)\big)$ for x, that is

$$x = \frac{1}{\exp(-r'')+1}\,. \tag{7.180}$$

   One also generates uniformly a random value of ϕ between 0 and 2π.

5. Set $x_{\oplus\ominus} = x$ and $\tilde{v}_\oplus = v$. Now we need to apply the last veto. Generate a random number $r_\oplus$ in the range

$$0 \le r_\oplus \le \left(N^{\oplus}_{q\bar{q}} + N_{g\bar{q}}\right) \frac{\alpha_{\rm S}}{\tilde{v}_\oplus} \frac{x_{\oplus\ominus}}{1 - x_{\oplus\ominus}} \,. \tag{7.181}$$

If

$$r_\oplus \le \frac{R^{\oplus}_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1, \Phi^{\oplus}_{\rm rad}\big) + R_{g\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1, \Phi^{\oplus}_{\rm rad}\big)}{B_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1\big)} \tag{7.182}$$

then accept the event. Otherwise, set $p_{\rm T}^{\rm max} = p_{\rm T}$ and go to step 2.

6. Call the $p_{\rm T}$ generated this way $p_{\rm T}^{\oplus}$. Set $p_{\rm T}^{\oplus} = 0$ in case the event is returned as a Born one. Repeat the previous steps for the second integral, that is set $N = N_\ominus$ and $\bar{x} = \bar{x}_\ominus$, with the obvious changes in step 5, that now becomes

   5$'$. Set $x_{\oplus\ominus} = x$ and $\tilde{v}_\ominus = v$ and generate a random number $r_\ominus$ in the range

   $$0 \le r_\ominus \le \left(N^{\ominus}_{q\bar{q}} + N_{qg}\right) \frac{\alpha_{\rm S}}{\tilde{v}_\ominus} \frac{x_{\oplus\ominus}}{1 - x_{\oplus\ominus}} \,. \tag{7.183}$$

   If

   $$r_\ominus \le \frac{R^{\ominus}_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1, \Phi^{\ominus}_{\rm rad}\big) + R_{qg}\big(\bar{\boldsymbol{\Phi}}_1, \Phi^{\ominus}_{\rm rad}\big)}{B_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1\big)} \tag{7.184}$$

   then accept the event. Otherwise, set $p_{\rm T}^{\rm max} = p_{\rm T}$ and go to step 2. Call the $p_{\rm T}$ generated this way $p_{\rm T}^{\ominus}$. Set $p_{\rm T}^{\ominus} = 0$ in case the event is returned as a Born one.

7. According to the highest-$p_{\rm T}$ bid method of appendix B, choose the highest value between $p_{\rm T}^{\oplus}$ and $p_{\rm T}^{\ominus}$. If both $p_{\rm T}^{\oplus}$ and $p_{\rm T}^{\ominus}$ are zero, then return the event as a Born-like one.

   - In case the $p_{\rm T}$ chosen is $p_{\rm T}^{\oplus}$, if

     $$r_\oplus > \frac{R^{\oplus}_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1, \Phi^{\oplus}_{\rm rad}\big)}{B_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1\big)} \,, \tag{7.185}$$

     then generate a $g\bar{q}$ event. Otherwise, generate a $q\bar{q}$ event with radiation emitted by the $\oplus$ incoming quark.

   - In case the $p_{\rm T}$ chosen is $p_{\rm T}^{\ominus}$, if

     $$r_\ominus > \frac{R^{\ominus}_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1, \Phi^{\ominus}_{\rm rad}\big)}{B_{q\bar{q}}\big(\bar{\boldsymbol{\Phi}}_1\big)} \,, \tag{7.186}$$

     then generate a qg event. Otherwise, generate a $q\bar{q}$ event with radiation emitted by the $\ominus$ incoming quark.

7.4 $h_\oplus h_\ominus \to V$ in the Frixione, Kunszt and Signer formalism

We now consider the case of the production of a massive vector V of mass M in hadron-hadron collisions, in the FKS formalism.

Radiation variables and inverse construction. We use the kinematics notation introduced in section 7, that is, the Born kinematics is characterized by the rapidity $\bar{Y}$ of the produced vector boson, and the phase space and momentum fractions are given by eqs. (7.98) and (7.99)

$$d\bar{\mathbf{\Phi}}_1 = \frac{2\pi}{S} d\bar{Y}\,, \qquad \bar{x}_\oplus = \sqrt{\frac{M^2}{S}} e^{\bar{Y}}, \qquad \bar{x}_\ominus = \sqrt{\frac{M^2}{S}} e^{-\bar{Y}}\,. \tag{7.187}$$

According to eqs. (7.105)–(7.107), the real-emission processes are characterized by two final-state momenta, k_1 (the V momentum) and k_2, the momentum of the radiated parton. The (only) radiated parton is the FKS parton. Following the prescriptions of section 5.1, we define

$$k_2^0 = \frac{\sqrt{s}}{2}\,\xi\,, \qquad k_2 = k_2^0\,(1, \sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)\,, \tag{7.188}$$

and we take ξ, $y = \cos\theta$ and ϕ as the radiation variables. The real-emission phase space is given by (see eqs. (5.9) and (5.10))

$$d\mathbf{\Phi}_2 = d\bar{\mathbf{\Phi}}_1\, d\Phi_{\rm rad}\,, \qquad d\Phi_{\rm rad} = \frac{s}{(4\pi)^3}\,\frac{\xi}{1-\xi}\,d\xi\,dy\,d\phi\,, \tag{7.189}$$

and the procedure described in section 5.1.2 allows one to fully reconstruct the two-body phase space, give the Born barred variable $\bar{Y}$ and the three radiation variables. In particular, the requirement that both $x_\oplus$ and $x_\ominus$ of eq. (5.11) be less than 1 forces ξ to be in the range $0 \le \xi \le \xi_{\max}$, where $\xi_{\max}$ is given by eq. (5.13).

Virtual corrections. The virtual corrections to the process can be found in ref. [38], eq. (82)

$$\mathcal{V}_{b,\,q\bar{q}} = \left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\,\frac{\alpha_{\rm S}}{2\pi}\,C_{\rm F}\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2\right]\mathcal{B}_{q\bar{q}}, \tag{7.190}$$

where we drop the argument $\bar{\mathbf{\Phi}}_1$ of the virtual and Born term, for ease of notation. Following section 7, we indicate with q the Born flavour index f_b, that carries information on the flavour of the (massless) incoming parton, along the $\oplus$ direction. Using a standard convention for the numbering scheme, we have $q = -5, \ldots, -1, 1, \ldots, 5$, and $\bar{q} = -q$. We notice that

$$\frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} = \frac{1}{\Gamma(1-\epsilon)} + \mathcal{O}\left(\epsilon^3\right), \tag{7.191}$$

so that, choosing $Q^2 = M^2$ in eq. (2.93), we get

$$\mathcal{V}_{b,\,q\bar{q}} = \mathcal{N}\,\frac{\alpha_{\rm S}}{2\pi}\,C_{\rm F}\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2\right]\mathcal{B}_{q\bar{q}}. \tag{7.192}$$

From eq. (2.92) we thus get

$$\mathcal{V}_{{\rm fin},\,q\bar{q}} = C_{\rm F}\left(\pi^2 - 8\right)\mathcal{B}_{q\bar{q}}\,. \tag{7.193}$$

Notice that the $\mathcal{B}_{ij}$ terms in (2.92) have only the $\mathcal{B}_{\oplus\ominus}$ and $\mathcal{B}_{\ominus\oplus}$ components in this case, and their contribution vanishes because we have chosen $Q^2 = M^2 = 2\,(k_\oplus \cdot k_\ominus)$. According to eqs. (2.97), (2.100) and (2.101) we have

$$\mathcal{B}_{\oplus\ominus} = \mathcal{B}_{\ominus\oplus} = C_{\rm F}\mathcal{B}_{q\bar{q}}\,, \tag{7.194}$$

$$\sum_{i,j\in\mathcal{I},\,i\neq j} \mathcal{I}_{ij}\,\mathcal{B}_{ij} = (\mathcal{B}_{\oplus\ominus}+\mathcal{B}_{\ominus\oplus})\,\mathcal{I}_{\oplus\ominus} = -(\mathcal{B}_{\oplus\ominus}+\mathcal{B}_{\ominus\oplus})\,{\rm Li}_2(1) = -\frac{\pi^2}{6}\,(\mathcal{B}_{\oplus\ominus}+\mathcal{B}_{\ominus\oplus})\,, \tag{7.195}$$

$$\mathcal{Q} = -3\,C_{\rm F}\,\log\frac{\mu_{\rm F}^2}{M^2}\,, \tag{7.196}$$

where we have chosen $\xi_c = 1$. Finally, eq. (2.99) gives

$$\mathcal{V}_{q\bar{q}} = \frac{\alpha_{\rm S}}{2\pi}\,C_{\rm F}\left(-3\,\log\frac{\mu_{\rm F}^2}{M^2} + \frac{2}{3}\,\pi^2 - 8\right)\mathcal{B}_{q\bar{q}}\,. \tag{7.197}$$

Collinear remnants. From eq. (2.102) we immediately get the collinear remnants

$$\mathcal{G}_{\oplus}^{q\bar{q}} = \frac{\alpha_{\rm S}}{2\pi}\,C_{\rm F}\left\{(1+z^2)\left[\left(\frac{1}{1-z}\right)_+\log\frac{M^2}{z\mu_{\rm F}^2}+2\left(\frac{\log(1-z)}{1-z}\right)_+\right]+1-z\right\}\mathcal{B}_{q\bar{q}}(zk_\oplus,k_\ominus)\,, \tag{7.198}$$

$$\mathcal{G}_{\oplus}^{g\bar{q}} = \frac{\alpha_{\rm S}}{2\pi}\,T_{\rm F}\left\{\left[z^2+(1-z)^2\right]\left[\log\frac{M^2}{z\mu_{\rm F}^2}+2\log(1-z)\right]+2z(1-z)\right\}\mathcal{B}_{q\bar{q}}(zk_\oplus,k_\ominus)\,, \tag{7.199}$$

and the corresponding ones for the $\ominus$ collinear direction, and where we have set $\delta_{\rm I} = 2$.

Real corrections. The real production contributions are given by eqs. (7.132)–(7.134) (see also [38])

$$\mathcal{R}_{q\bar{q}} = 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{2\,s}\left[\frac{u}{t}+\frac{t}{u}+\frac{2\,s\,M^2}{tu}\right], \tag{7.200}$$

$$\mathcal{R}_{g\bar{q}} = -2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{2\,s}\left[\frac{u}{s}+\frac{s}{u}+\frac{2\,t\,M^2}{su}\right], \tag{7.201}$$

(and the analogous one for $\mathcal{R}_{qg}$) where, according to the definitions given in eqs. (7.108)–(7.110),

$$s = \frac{M^2}{1-\xi}\,,\qquad t = -\frac{s}{2}\,\xi\,(1+y)\,,\qquad u = -\frac{s}{2}\,\xi\,(1-y)\,. \tag{7.202}$$

In terms of the FKS variables we have

$$\mathcal{R}_{q\bar{q}} = 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{2\,s}\left\{2\,\frac{1+y^2}{1-y^2}+\frac{8(1-\xi)}{\xi^2\,(1-y^2)}\right\}, \tag{7.203}$$

$$\mathcal{R}_{g\bar{q}} = 2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{2\,s}\left\{\frac{2[1-\xi(1-\xi)(1+y)]}{\xi(1-y)}+\frac{\xi(1-y)}{2}\right\}, \tag{7.204}$$

so that $(1-y^2)\,\xi^2\,\mathcal{R}$ are regular functions. Using eq. (2.84) we obtain immediately

$$\hat{\mathcal{R}}_{q\bar{q}} = 2\,\frac{C_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{2\,s}\left[2\left(1+y^2\right)\xi^2+8(1-\xi)\right]\left\{\frac{1}{2}\left(\frac{1}{\xi}\right)_+\left[\left(\frac{1}{1-y}\right)_+ +\left(\frac{1}{1+y}\right)_+\right]\right\}\frac{1}{\xi}\,, \tag{7.205}$$

$$\hat{\mathcal{R}}_{g\bar{q}} = 2\,\frac{T_{\rm F}}{N_c}\,g^2\,g_s^2\,\frac{1}{2\,s}\left\{2[1-\xi(1-\xi)(1+y)]+\frac{\xi^2(1-y)^2}{2}\right\}\left(\frac{1}{1-y}\right)_+\frac{1}{\xi}\,, \tag{7.206}$$

with the + distributions defined in eq. (2.90).

Generation of the Born variable $\bar{\boldsymbol{Y}}$. The expression for $\bar{B}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1)$ of eq. (4.13) is then given by

$$\begin{aligned}\bar{B}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1) = {} & B_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1) + V_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1) \\ & + \int d\Phi_{\rm rad} \left[\hat{R}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{\boldsymbol{1}}, \Phi_{\rm rad}) + \hat{R}_{qg}(\bar{\boldsymbol{\Phi}}_{\boldsymbol{1}}, \Phi_{\rm rad}) + \hat{R}_{g\bar{q}}(\bar{\boldsymbol{\Phi}}_{\boldsymbol{1}}, \Phi_{\rm rad})\right] \\ & + \int_{\bar{x}_\oplus}^1 \frac{dz}{z} \left[G_\oplus^{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\oplus}) + G_\oplus^{g\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\oplus})\right] + \int_{\bar{x}_\ominus}^1 \frac{dz}{z} \left[G_\ominus^{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\ominus}) + G_\ominus^{qg}(\bar{\boldsymbol{\Phi}}_{1,\ominus})\right], \end{aligned} \tag{7.207}$$

where

$$B_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1) = \mathcal{B}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1)\ \mathcal{L}_{q\bar{q}}(\bar{x}_\oplus, \bar{x}_\ominus), \tag{7.208}$$

$$V_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1) = \mathcal{V}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1)\ \mathcal{L}_{q\bar{q}}(\bar{x}_\oplus, \bar{x}_\ominus), \tag{7.209}$$

$$\hat{R}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1, \Phi_{\rm rad}) = \hat{\mathcal{R}}_{q\bar{q}}(\bar{\boldsymbol{\Phi}}_1, \Phi_{\rm rad})\, \mathcal{L}_{q\bar{q}}(x_\oplus, x_\ominus), \tag{7.210}$$

$$\hat{R}_{qg}(\bar{\boldsymbol{\Phi}}_1, \Phi_{\rm rad}) = \hat{\mathcal{R}}_{qg}(\bar{\boldsymbol{\Phi}}_1, \Phi_{\rm rad})\, \mathcal{L}_{qg}(x_\oplus, x_\ominus), \tag{7.211}$$

$$\hat{R}_{g\bar{q}}(\bar{\boldsymbol{\Phi}}_1, \Phi_{\rm rad}) = \hat{\mathcal{R}}_{g\bar{q}}(\bar{\boldsymbol{\Phi}}_1, \Phi_{\rm rad})\, \mathcal{L}_{g\bar{q}}(x_\oplus, x_\ominus), \tag{7.212}$$

$$G_\oplus^{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\oplus}) = \mathcal{G}_\oplus^{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\oplus})\, \mathcal{L}_{q\bar{q}}\left(\frac{\bar{x}_\oplus}{z}, \bar{x}_\ominus\right), \tag{7.213}$$

$$G_\oplus^{g\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\oplus}) = \mathcal{G}_\oplus^{g\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\oplus})\, \mathcal{L}_{g\bar{q}}\left(\frac{\bar{x}_\oplus}{z}, \bar{x}_\ominus\right), \tag{7.214}$$

$$G_\ominus^{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\ominus}) = \mathcal{G}_\ominus^{q\bar{q}}(\bar{\boldsymbol{\Phi}}_{1,\ominus})\, \mathcal{L}_{q\bar{q}}\left(\bar{x}_\oplus, \frac{\bar{x}_\ominus}{z}\right), \tag{7.215}$$

$$G_\ominus^{qg}(\bar{\boldsymbol{\Phi}}_{1,\ominus}) = \mathcal{G}_\ominus^{qg}(\bar{\boldsymbol{\Phi}}_{1,\ominus})\, \mathcal{L}_{qg}\left(\bar{x}_\oplus, \frac{\bar{x}_\ominus}{z}\right), \tag{7.216}$$

with $x_\oplus, x_\ominus$ given in eq. (5.11) and the luminosity $\mathcal{L}$ defined in eq. (7.147).

As explained in section 5.1.1, the integration range in the radiation variable ξ is limited by the condition (5.12)

$$0 \le \xi \le \xi_{\rm max}\,. \tag{7.217}$$

Making the change of variable

$$1 - \xi = z\,, \tag{7.218}$$

the integration limits become

$$z_{\rm min}(y) \le z \le 1\,, \tag{7.219}$$

where

$$z_{\rm min}(y) = \max\left\{\frac{2(1+y)\,\bar{x}_\oplus^2}{\sqrt{(1+\bar{x}_\oplus^2)^2(1-y)^2 + 16\,y\,\bar{x}_\oplus^2} + (1-y)(1-\bar{x}_\oplus^2)}, \frac{2(1-y)\,\bar{x}_\ominus^2}{\sqrt{(1+\bar{x}_\ominus^2)^2(1+y)^2 - 16\,y\,\bar{x}_\ominus^2} + (1+y)(1-\bar{x}_\ominus^2)}\right\}. \tag{7.220}$$

It is convenient to introduce a rescaled variable $\tilde{z}$

$$\tilde{z} = \frac{z - z_{\rm min}(y)}{1 - z_{\rm min}(y)}\,, \qquad z = z_{\rm min}(y) + \tilde{z}\,\left(1 - z_{\rm min}(y)\right), \tag{7.221}$$

and use the identity

$$\left(\frac{1}{\xi}\right)_+ = \left(\frac{1}{1-z}\right)_+ = \frac{1}{1-z_{\min}(y)} \left\{ \left(\frac{1}{1-\tilde{z}}\right)_+ + \delta(1-\tilde{z}) \log[1-z_{\min}(y)] \right\} , \tag{7.222}$$

in order to simplify the treatment of the + distribution in the integration of $\hat{R}$.

The integral of the distributions requires some care. We follow, as an example, the integral of the $1/(1-y)_+$ contribution in the $\hat{R}_{q\bar{q}}$ term. The integral has the form

$$I = \int_{-1}^{1} dy \int_{1-z_{\min}(y)}^{1} d\xi \left(\frac{1}{1-y}\right)_+ \left(\frac{1}{\xi}\right)_+ f(\xi, y) \,. \tag{7.223}$$

We change variable from ξ to $\tilde{z}$ using eq. (7.222), and obtain

$$\begin{aligned} I =& \int_{-1}^{1} dy \int_{0}^{1} d\tilde{z} \left(\frac{1}{1-y}\right)_+ \left(\frac{1}{1-\tilde{z}}\right)_+ f(\xi(\tilde{z},y)\,,y) \\ &+ \int_{-1}^{1} dy \left(\frac{1}{1-y}\right)_+ \log\left[1-z_{\min}(y)\right] f(0,y) \\ =& \int_{-1}^{1} dy \int_{0}^{1} d\tilde{z} \, \frac{1}{1-y} \left[\frac{f(\xi(\tilde{z},y),y) - f(0,y)}{1-\tilde{z}} - \frac{f(\xi(\tilde{z},1),1) - f(0,1)}{1-\tilde{z}} \right] \\ &+ \int_{-1}^{1} dy \, \frac{1}{1-y} \left\{ \log\left[1-z_{\min}(y)\right] f(0,y) - \log\left[1-z_{\min}(1)\right] f(0,1) \right\}, \end{aligned} \tag{7.224}$$

that is manifestly finite and can be computed numerically.

The collinear contributions also have a restricted range in the z variable. Considering for simplicity only the $\mathcal{G}_{\oplus}^{q\bar{q}}$ term, we have $\bar{x}_{\oplus} \le z \le 1$, so that, as usual, we need to make use of the identities

$$\int_{\bar{x}_{\oplus}}^{1} dz \left(\frac{1}{1-z}\right)_+ f(z) = \log(1-\bar{x}_{\oplus}) f(1) + \int_{\bar{x}_{\oplus}}^{1} dz \, \frac{f(z)-f(1)}{1-z} \,, \tag{7.225}$$

$$\int_{\bar{x}_{\oplus}}^{1} dz \left(\frac{\log(1-z)}{1-z}\right)_+ f(z) = \frac{1}{2} \log^2(1-\bar{x}_{\oplus}) f(1) + \int_{\bar{x}_{\oplus}}^{1} dz \, \frac{\log(1-z)[f(z)-f(1)]}{1-z}, \tag{7.226}$$

and also in this case we define a rescaled variable $0 \le \tilde{z} \le 1$

$$\tilde{z} = \frac{z-\bar{x}_{\oplus}}{1-\bar{x}_{\oplus}} \,, \qquad\qquad z = \bar{x}_{\oplus} + \tilde{z}(1-\bar{x}_{\oplus}) \,, \tag{7.227}$$

and rewrite the z integrations as $\tilde{z}$ integrations. At the end of this procedure, the most general form one can obtain for $\bar{B}$ is

$$\begin{aligned} \bar{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) =& B^{a}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) + \int_{0}^{1} d\tilde{z} \, B^{b}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, \tilde{z}) + \int_{-1}^{1} dy \int_{0}^{2\pi} d\phi \, B^{c}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, y, \phi) \\ &+ \int_{-1}^{1} dy \int_{0}^{1} d\tilde{z} \int_{0}^{2\pi} d\phi \, B^{d}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1, \tilde{z}, y, \phi) \ . \end{aligned} \tag{7.228}$$

In order to perform numerically the generation of the Born term, we introduce the function (see eq. (4.24))

$$\tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\tilde{z},y,\phi) = \frac{1}{4\pi}B^a_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) + \frac{1}{4\pi}\int_0^1 d\tilde{z}\, B^b_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\tilde{z}) + \int_{-1}^1 dy \int_0^{2\pi} d\phi\, B^c_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,y,\phi) + \int_{-1}^1 dy \int_0^1 d\tilde{z} \int_0^{2\pi} d\phi\, B^d_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\tilde{z},y,\phi)\,, \tag{7.229}$$

so that

$$\bar{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1) = \int_{-1}^1 dy \int_0^1 d\tilde{z} \int_0^{2\pi} d\phi\, \tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\tilde{z},y,\phi)\ . \tag{7.230}$$

The generation of the Born variable $\bar{\mathbf{\Phi}}_1 = \{\bar{Y}\}$ is performed by using an integrator-unweighter program (like for example the BASES-SPRING program), that after a single four-dimensional integration of the function

$$\tilde{B}(\bar{\mathbf{\Phi}}_1,\tilde{z},y,\phi) = \sum_q \tilde{B}_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\tilde{z},y,\phi) \tag{7.231}$$

in the variables $\bar{Y},\tilde{z},y,\phi$, can generate 4-tuples of $\bar{Y},\tilde{z},y,\phi$ values, distributed according to the weight $\tilde{B}(\bar{Y},\tilde{z},y,\phi)$. For each generated 4-tuple, we also generate q with a probability proportional to $\tilde{B}_{q\bar{q}}(\bar{Y},\tilde{z},y,\phi)$. We only keep the $\bar{Y}$ and q generated value, and neglect $y,\tilde{z},\phi$, which corresponds to integrating over them.

As already noticed at the end of section 7, observe that $\tilde{B}(\bar{Y},\tilde{z},y,\phi)$ is positive, unless the $\mathcal{O}(\alpha_S)$ terms overcome the Born term. If this happens, the whole perturbative approach breaks down. This can happen if M is too small, or if we are in extreme regions of phase space, like the threshold region (i.e. when the values of $\bar{\mathbf{\Phi}}_1$ and M are forcing $\bar{x}_\oplus$ or $\bar{x}_\ominus$ to 1). In practical applications it is wise to check that the fraction of negative weight in the integral of $\tilde{B}(\bar{Y},\tilde{z},y,\phi)$ is small, and can be safely neglected.

Generation of the radiation variables. We assume that we have generated $\bar{\mathbf{\Phi}}_1$ and q according to the procedure given above. The Sudakov form factor for the generation of radiation, according to eq. (4.16) is given by

$$\Delta_q(\bar{\mathbf{\Phi}}_1,p_T) = \exp\left\{-\int d\Phi_{\rm rad}\,\frac{R_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}) + R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}) + R_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)}\,\theta(k_T-p_T)\right\}, \tag{7.232}$$

where

$$k_T^2 = \frac{s}{4}\,\xi^2\left(1-y^2\right) = \frac{M^2}{4(1-\xi)}\,\xi^2\left(1-y^2\right) \tag{7.233}$$

is the exact squared transverse momentum of the radiated parton. The factorization and renormalization scales in eq. (7.232) should be taken equal to k_T^2. In order to generate the radiation variables, we use the hit-and-miss technique introduced in section 4.4.2. We need to find a simple upper bound for the expression

$$\frac{s}{(4\pi)^3}\,\frac{\xi}{1-\xi}\,\frac{R_{q\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}) + R_{qg}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad}) + R_{g\bar{q}}(\bar{\mathbf{\Phi}}_1,\Phi_{\rm rad})}{B_{q\bar{q}}(\bar{\mathbf{\Phi}}_1)}\,, \tag{7.234}$$

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in order to perform the generation of radiation using the veto method. With the same considerations of section 4.4.2 and using the upper bounds for the parton distribution functions found in appendix D, it is easy to show that the upper bounding function found in ref. [22]

$$U_q = N_q \, \frac{\alpha_S\left(k_T^2\right)}{\xi\,(1-y^2)} \tag{7.235}$$

is suited also to the present case.

When implementing the bound (7.235) in practice, the whole phase space is searched randomly in order to determine the bound normalization N_q. At this stage, problems can arise, due to the fact that available parton-density parametrization do not necessarily comply with the bound (D.8). One simple example is the case of the b quark density, that is set arbitrarily to zero above a given value of x, generally smaller than the value of x at which the gluon density is set to zero. These regions give very small contribution to the cross section, and must be carefully cut away, unless one is using a pdf parametrization that is fully consistent with evolution also in the very large-x region.

The bound (7.235) has been obtained assuming that we are not near the small-x region. If this is the case, the bound may become inefficient. We observe that, in case the parton distribution functions obey Feynman scaling (i.e. $f(x) \approx 1/x$ at small x), the bound is still adequate in the small-x region. Modern pdf parametrizations, however, prefer the faster rising behavior $f(x) \approx 1/x^\delta$, with $\delta > 1$. In this case, the bound may become inefficient at small x, and numerical tricks (like, for example, dividing the integration range in small intervals and adopting different N_q values in each interval) should be used to overcome this problem.

The generation of the event according to the bound (7.235) is documented in great detail in appendix D of ref. [22], and we do not repeat it here.

8. Conclusions

In this work we have presented in full details the POWHEG method for interfacing NLO calculations to parton shower. The purpose of this paper, besides providing all the technical details needed for the implementation of the method, was also to demonstrate its full generality. We have given detailed formulae in the case of cross sections that do not involve massive coloured partons. The extension to this case is straightforward both in the FKS and in the CS framework. We have not tried to include it in our discussion, in order not to increase the complexity of the presentation.

We believe that the method presented here can be applied to a large number of interesting LHC processes, and that authors of QCD calculation should be able to build their own implementation with only a modest effort. In order to ease this task, we have collected POWHEG software in a repository at the location: http://moby.mib.infn.it/~nason/POWHEG/FNOpaper/. Among other things, the software for the implementation of the inverse mapping in FKS, the program `mint` (see ref. [39]) for the integration and generation of unweighted events, and the full implementation of the e^+e^- example in the FKS approach can be found there.

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A. The veto technique

In this section we describe a method to generate a set of d variables x, distributed according to

$$f(x)\,\Delta(h(x)) \tag{A.1}$$

where

$$\Delta(h) = \exp\left\{-\int d^d x'\, f\big(x'\big)\ \theta\big(h\big(x'\big)-h\big)\right\}. \tag{A.2}$$

We assume, as usual, that f and h are non-negative functions, and that the unrestricted integral of f is divergent, that is

$$\Delta(0) = \exp\left\{-\int d^d x'\, f\big(x'\big)\right\} = 0\,. \tag{A.3}$$

With these assumptions, upon multiplying the infinitesimal probability $f(x)\,\Delta(h(x))\,d^d x$ by $\delta(h-h(x))\,dh$, we can integrate over $d^d x$ and interpret it as infinitesimal probability for the variable h

$$dh\int d^d x\,\delta(h-h(x))\,f(x)\exp\left\{-\int d^d x'\, f\big(x'\big)\ \theta\big(h\big(x'\big)-h\big)\right\} = dh\,\frac{d\Delta(h)}{dh} = d\Delta(h)\,, \tag{A.4}$$

that shows that the probability is uniform in $\Delta(h)$. In principle, the generation of events is therefore straightforward: one generates a uniform random number r between 0 and 1, and solves the equation $\Delta(H) = r$ for H (here we have used the fact that $\Delta(0) = 0$). At this point, the variables x have distribution function equal to

$$\delta(H-h(x))\,f(x)\exp\left\{-\int d^d x'\, f\big(x'\big)\ \theta\big(h\big(x'\big)-H\big)\right\}, \tag{A.5}$$

where the exponent is now just a number (a normalization factor), so that the variables x are on the surface $\delta(H-h(x))$ with a distribution function proportional to $f(x)\,\delta(H-h(x))$. The generation of these variables can be done, for example, with a hit-and-miss technique, or, if the integration can be performed analytically, by generating $(d-1)$ random numbers r_i, uniformly distributed between 0 and 1, and solving

$$\int_{x_{i0}}^{X_i} dx_i \prod_{k\neq i}\int dx_k\,\delta(H-h(x))\,f(x) = r_i \tag{A.6}$$

for X_i, where x_{i0} is the lower limit for the variable x_i, and all the other variables are integrated over their full range of validity.

In practise, however, the solution of the equation $\Delta(H) = r$ is, in most cases, very heavy, from a numerical point of view. This difficulty can be overcome by means of the so-called veto method. We assume that there is a function $F(x) \geq f(x)$ for all x values, and that

$$\Delta_F(h) = \exp\left[-\int d^d x'\, F(x')\; \theta\big(h\,(x') - h\big)\right] \tag{A.7}$$

has a simple form, so that the solution of the equation $\Delta_F(H) = r$ and the generation of the distribution $F(x)\,\delta(H - h(x))$ are reasonably simple. Then, we implement the following procedure:

1. Set $H_{\rm max}$ equal to the maximum allowed value, such that $\Delta_F(H_{\rm max}) = 1$.
2. Generate a flat random number $0 < r < 1$ and solve the equation
$$\frac{\Delta_F(H)}{\Delta_F(H_{\rm max})} = r \tag{A.8}$$
for H (a solution with $0 < H < H_{\rm max}$ always exists for $0 < r < 1$).
3. Generate x according to $F(x)\,\delta(h(x) - H)$.
4. Generate a new random number r'.
5. If $r' > f(x)/F(x)$ then the event is vetoed, we set $H_{\rm max} = H$, go to step 2 and continue. Otherwise the event is accepted, and the procedure stops.

The resulting events are distributed according to eq. (A.1). The proof of this statement is simple but non-trivial, and is given in appendix C of ref. [22].

B. The highest-$p_{\rm T}$ bid procedure

Our aim is to generate $(k,\, x_k)$ pairs with a probability

$$f_k(x_k) \prod_i \Delta_i(h_k(x_k))\; d^d x_k\,, \tag{B.1}$$

where

$$\Delta_i(h) = \exp\left\{-\int d^d x'_i\, f_i(x'_i)\; \theta\big(h_i(x'_i) - h\big)\right\}. \tag{B.2}$$

We assume, as usual, that the f_i and h_i are non-negative functions, and that the unrestricted integral of the f_i is divergent, that is

$$\Delta_i(0) = \exp\left\{-\int d^d x'_i\, f_i(x'_i)\right\} = 0\,. \tag{B.3}$$

Under these conditions, we have the identity

$$\int d^d x_k\, \Delta_k(h_k(x_k))\; f_k(x_k)\, \delta(h_k(x_k) - h) = \frac{d}{dh}\Delta_k(h)\,, \tag{B.4}$$

so that

$$\int d^d x_k \, \Delta_k(h_k(x_k)) \, f_k(x_k) \, \theta(h - h_k(x_k))$$
$$= \int_0^\infty dh' \int d^d x_k \, \Delta_k(h_k(x_k)) \, f_k(x_k) \, \delta(h' - h_k(x_k)) \, \theta\big(h - h'\big)$$
$$= \int_0^\infty dh' \, \frac{d}{dh'} \Delta_k\big(h'\big) \, \theta\big(h - h'\big) = \int_0^h dh' \, \frac{d}{dh'} \Delta_k\big(h'\big) = \Delta_k(h) \,, \quad \text{(B.5)}$$

where we have used the fact that $\Delta_k(0) = 0$. If we interpret h and h_k as transverse momenta, then $\Delta_k(h)$ in eq. (B.5) corresponds to the probability of not emitting a parton with transverse momentum bigger than h.

The procedure to generate the distribution in eq. (B.1), using the highest-bid method, is the following. For each k, we generate an x_k value with probability

$$\Delta_k(h_k(x_k)) \, f_k(x_k) \, d^d x_k \,, \quad \text{(B.6)}$$

as described in appendix A, and then we pick the k value with the largest $h_k(x_k)$. In fact, the probability that the generated $(k, \, x_k)$ has the largest $h_k(x_k)$ is precisely given by the product of its generation probability (eq. (B.6)) times the probability that all the other $h_i(x_i)$ are less then $h_k(x_k)$, which is given by the product

$$\prod_{i \neq k} \Delta_i(h_k(x_k)) \,, \quad \text{(B.7)}$$

and together they reconstruct eq. (B.1).

C. Soft angular integrals

The soft angular integrals eqs. (4.46) and (4.47) can all be obtained from the basic integral

$$I\big(k_i, v, k^0\big) = \int d\Omega \frac{k_i \cdot v}{(k_i \cdot k)(k \cdot v)}, \quad \text{(C.1)}$$

where v is an arbitrary timelike vector and $d\Omega$ is the element of the solid angle of k with respect to a reference direction. First we notice that, since the integrand scales like $(k^0)^{-2}$,

$$\int d\Omega \int_{f(\Omega)}^{ef(\Omega)} k^0 dk^0 \frac{k_i \cdot v}{(k_i \cdot k)(k \cdot v)} = I(k_i, v, 1), \quad \text{(C.2)}$$

(where e is the Euler constant) independent of f, for any arbitrary function $f(\Omega)$. We can also write

$$I(k_i, v, 1) = \int \frac{d^3 k}{k^0} \frac{k_i \cdot v}{(k_i \cdot k)(k \cdot v)} \theta\big(k^0 - f(\Omega)\big) \; \theta\big(ef(\Omega) - k^0\big) \,, \quad \text{(C.3)}$$

and perform a change of variables, that is also a Lorentz transformation, $k \to \Lambda k$, to obtain

$$I(k_i, v, 1) = \int \frac{d^3 k}{k^0} \frac{k_i' \cdot v'}{(k_i' \cdot k)(k \cdot v')} \theta\big(k^0 - f'(\Omega)\big) \; \theta\big(ef'(\Omega) - k^0\big) \,, \quad \text{(C.4)}$$

where $v' = \Lambda^{-1} v$ and $k_i' = \Lambda^{-1} k_i$. The function f also undergoes a change, and becomes f'. But we have just shown that the integral does not depend upon f, so that we conclude

$$I(k_i, v, 1) = I(k_i', v', 1). \tag{C.5}$$

We thus perform the integral in a frame where v has only the time component. We get

$$I(k_i, v, k^0) = \frac{2\pi}{k_0^2} \int d\cos\theta \, \frac{1}{1-\beta\cos\theta} = \frac{2\pi}{\beta\, k_0^2} \log\frac{1+\beta}{1-\beta}\,, \qquad \beta = \sqrt{1 - \frac{k_i^2\, v^2}{(k\cdot v)^2}}\,. \tag{C.6}$$

Setting $v = k_{ij}$ in eq. (C.6) yields immediately eq. (4.47). For small k_i^2, eq. (C.6) becomes

$$I(k_i, v, k^0) = \frac{2\pi}{k_0^2} \log\frac{4(k_i\cdot v)^2}{k_i^2\, v^2}\,. \tag{C.7}$$

Taking the difference of the above formula with $v = k_{ij}$ and $v = n$ we get (4.46).

D. Parton-distribution-function upper bounds

In this section, we discuss the bounds on the parton-distribution-function (pdf) ratios of the form

$$\frac{f_g(x/z, \mu_{\rm F}^2)}{f_g(x, \mu_{\rm F}^2)}\,, \qquad \frac{f_q(x/z, \mu_{\rm F}^2)}{f_q(x, \mu_{\rm F}^2)}\,, \qquad \frac{f_g(x/z, \mu_{\rm F}^2)}{f_q(x, \mu_{\rm F}^2)}\,, \qquad \frac{f_q(x/z, \mu_{\rm F}^2)}{f_g(x, \mu_{\rm F}^2)}\,, \tag{D.1}$$

where q stands for a quark or an antiquark, and $x \le z \le 1$. We begin by noticing that the first two ratios are always bounded by a number of order 1, because the pdf's are never fast growing functions of their argument. In fact, only the pdf of the valence quarks grows mildly for moderate x values. In the remaining two cases, the ratio of parton densities is between different parton species, and must be discussed with care. We consider first the g/q case. First of all, since the gluon pdf is a monotonic decreasing function

$$f_g(x/z, \mu_{\rm F}^2) < f_g(x, \mu_{\rm F}^2)\,, \tag{D.2}$$

so that we only need a bound for

$$\frac{f_g(x, \mu_{\rm F}^2)}{f_q(x, \mu_{\rm F}^2)}\,. \tag{D.3}$$

At small values of x, this ratio is always bounded, because the gluon and quark densities have similar behavior in the small-x limit. For large x, if q is a valence quark, the ratio is also bounded, since the gluon is generated by valence quarks through evolution. In case q is a sea quark, the corresponding density may be softer than the gluon density. In the worst case, however, the sea quark is generated by the gluon through evolution. Assuming that parton densities behave as a power of $1-x$ at large x,

$$f_g(x, \mu_{\rm F}^2) \sim (1-x)^\delta\,, \tag{D.4}$$

the Altarelli-Parisi equation in the large-x limit yields

$$\mu^2 \frac{d f_q(x, \mu_{\rm F}^2)}{d\mu_{\rm F}^2} \sim \frac{T_F \alpha_{\rm S}}{2\pi} \int_x^1 \frac{dz}{z} (1-z)^\delta \sim \frac{T_F \alpha_{\rm S}}{2\pi} (1-x)^{\delta+1}\,, \tag{D.5}$$

and therefore

$$\frac{f_g(x,\mu_{\rm F}^2)}{f_q(x,\mu_{\rm F}^2)} \lesssim \frac{C}{1-x}\,. \tag{D.6}$$

Since

$$\frac{1}{1-x} < \frac{1}{1-z}\,, \tag{D.7}$$

we conclude that

$$\frac{f_g(x/z,\mu_{\rm F}^2)}{f_q(x,\mu_{\rm F}^2)} \leq \frac{C}{1-z}\,. \tag{D.8}$$

For the q/g case, since

$$f_q(x/z,\mu_{\rm F}^2) \lesssim f_q(x,\mu_{\rm F}^2)\,, \tag{D.9}$$

we need a bound for

$$\frac{f_q(x,\mu_{\rm F}^2)}{f_g(x,\mu_{\rm F}^2)}\,. \tag{D.10}$$

The symbol $\lesssim$ in eq. (D.9) is to be interpreted as "almost everywhere smaller than". In fact, if q is a valence quark, its pdf can grow mildly in an intermediate x range. If q is not a valence quark, the ratio (D.10) is certainly small. We should only worry about the case when q is a valence quark. In this case, we assume for the quark a large x behavior of the form

$$f_q(x,\mu_{\rm F}^2) \sim (1-x)^\delta\,, \tag{D.11}$$

and, with a reasoning similar to the one used before, we can conclude that

$$\frac{f_q(x/z,\mu_{\rm F}^2)}{f_g(x,\mu_{\rm F}^2)} \leq \frac{C}{1-z}\,. \tag{D.12}$$

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